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WENTWORTH AND HUDSON

Exercise Manuals

No. II.

ALGEBRA

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WENTWORTH & HILL'S

EXERCISE MANUALS.

No. II.

ALGEBRA.

BOSTON, U.S.A.:
PUBLISHED BY GINN & COMPANY.
1892.

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PREFACE.

THIS book has been prepared to meet a generally-expressed wish for more problems than are found in most text-books on Algebra. It contains about 4500 problems, classified by subjects, and including all the topics usually taught in schools. The exercises have been selected mainly from the rich French collection of Morf and Tzaut. The simplest and the most difficult of the 12,000 problems in that collection have been rejected, and it is believed that these exercises will be not only new to American teachers but well-adapted to their wants.

The book may be used in connection with any Algebra for supplementary or review work; and, in order that it may be brought within the reach of scholars as well as teachers, the price has been made as low as possible.

G. A WENTWORTH.
G. A. HILL.

MAY, 1884.

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EXERCISE MANUAL.

ALGEBRA.

Ex. 1.

NOTATION.

1. Write the sum of a and b.
2. Increase a by 8 units.
3. What is the value of m, if $m = n + p$, and $n = 27$; and $p = 36$?
4. If z is a whole number, what is the whole number which immediately follows it?
5. John takes d steps forward, and then f steps backward; how many steps does he take in all?
6. To-day the thermometer indicates m degrees above 0; yesterday it indicated n degrees below 0; how many degrees warmer is it to-day than yesterday?
7. A man sells a horse for c dollars, and loses a dollars; what is the price paid for the horse?
8. If a is a whole number, what whole number immediately precedes it? and what precedes this last?
9. The sum of two numbers is m, and one of them is 18; what is the other?
10. What number does x have when (i.) $x+5=12$? (ii.) $x+37=63$: $x+n=18$: $x+m=p$?
11. A man bought a house for p dollars, and sold it for n dollars; how much did he gain?

12. A man sold a horse for c dollars, and gained 25 dollars; what did he pay for the horse?

13. A man has an income of m dollars; his expenses amount to n dollars; how much has he left (i.) if m is greater than n? (ii.) if m equals n? (iii.) if n is greater than m?

14. A man makes m steps forward, and then n steps backward; at what distance from the point of starting is he?

15. A body moves a feet in one direction, and then b feet in the opposite direction; at what distance from the point of starting is it if $a > b$? $a = b$? $a < b$?

16. Three towns, A, B, and C, are situated in a straight line. The distance from A to B is a miles, that from A to C is b miles; what is the distance from B to C?

17. If the length of a day is s hours, what is the length of the night?

18. Two steamboats move in the same direction, one behind the other. The first goes m yards per minute, and the second n yards. At the end of one minute, what distance separates them when $m > n$? when $n > m$?

19. A steamboat makes, without taking account of the current of the river and the force of the wind, 491 yards per minute. The current, by itself alone, will carry the boat 71 yards per minute, and the wind 100 yards. What distance will the boat go in one minute (i.) when it goes with the current and with the wind? (ii.) when it goes with the current but against the wind? (iii.) when it goes against the current but with the wind? (iv.) when it goes against the current and against the wind?

20. What will the results of the preceding problem become if d, s, and w are substituted for 491, 71, and 100, respectively?

21. What is the product if the multiplier is 7 and the multiplicand is a? if the multiplier is p and the multiplicand q?

22. A rectangular field is a rods long and b rods wide; what is its area?

23. A man pays m dollars for 1 yard of cloth; what must he pay for n yards?

24. A locomotive passes over a yards per second; how many yards will it pass over in t seconds?

25. A man has oats enough for n horses during t days; how long will they last 1 horse?

26. A product is p, the multiplicand is m; what is the multiplier?

27. What is the value of y when $9y = 63$? when $8y = 72$? when $43y = 2451$? when $53y = 1537$?

28. A body moves s yards in t seconds; how many yards does it move in 1 second?

29. How many times greater is the velocity of a bullet making m yards per second, than that of a stone making n yards per second?

30. What value has x when $x \div 7 = 9$? when $x \div 23 = 17$? when $35 \div x = 7$? when $703 \div x = 19$?

31. How many rolls of paper, g feet long by h feet wide, will be required to paper a wall a feet long by b feet high?

32. The cogs of 2 wheels fit into one another; the large wheel has n cogs, and the small wheel has m cogs. How many revolutions will the small wheel make when the large wheel has made 1 revolution?

33. Find the values of 3^2; 4^3; 2^{10}; 10^2; 2^4; 4^3.

34. Express the fifth power of 4, the xth power of y, the mth power of n, the pth power of q, the $x+y$th power of a?

35. If $x^y = z$, and $x = 5$, and $y = 7$, what is the value of z?

36. What is the shortest way of representing the product of 7 factors, each equal to 3, and what is its value?

37. What is the shortest way of representing
$a \times a \times a \times a \times a \times a \times a \times a \times a \times a \times a \times a$?

38. What is the shortest way of representing the product of x factors, each equal to m?

39. Into how many parts is a line divided, when it is first divided into m parts, and afterwards each of these m parts is divided into m new parts, and so on x times?

40. A man mixes a drop of a certain liquid with 4 drops of water; then he takes a drop of this mixture and mixes it again with 4 drops of water, and so on 6 times. What part of the first drop will finally remain in a drop of this mixture?

Ex. 2.

Addition.

Add:

1. $a^3 + 3a^2b + 3ab^2 + b^3$; $a^3 - 3a^2b + 3ab^2 - b^3$;
$-a^3 + 3a^2b - 3ab^2 + b^3$; $-a^3 - 3a^2b - 3ab^2 - b^3$;
$a^3 + 6a^2b + 6ab^2 + b^3$.

2. $x^4 + 3ax^3 - bx^2 + 3cx - d$; $4x^4 - 6ax^3 + 5bx^2 - 3cx + 2d$;
$-5x^4 - 6ax^3 - 5bx^2 - 3cx - 2d$;
$6x^4 + 7ax^3 - 8bx^2 + 6cx + 7d$;
$-10x^4 - ax^3 + 10bx^2 - cx + d$;
$-x^4 - ax^3 - bx^2 - cx - d$.

3. $a^5+5a^4b+6a^3b^2c-7ab$; $6a^5-a^4b-6a^3b^2c+10ab$;
$-2a^5+4a^4b+12a^3b^2c-10ab$;
$5a^5-16a^4b-11a^3b^2c+13ab$;
$-10a^5+8a^4b+a^3b^2c-6ab$.

4. $-7ax-5a\sqrt{x}+12y$; $5ax+a\sqrt{x}-8y$; $-ax+9a\sqrt{x}-y$;
$4ax-3a\sqrt{x}+5y$; $-ax+a\sqrt{x}+y$.

5. $a^3+\sqrt{x}-10+2\sqrt{ax}$; $2a^3-3\sqrt{x}+20+\sqrt{ax}$;
$-5a^3-3\sqrt{x}+30-5\sqrt{ax}$;
$-4a^3+2\sqrt{x}+1+12\sqrt{ax}$;
$-25a^3-25\sqrt{x}-25-25\sqrt{ax}$.

6. $a^m-2b^n+3c^p-4d^q$; $23a^m+16b^n-13c^p+12d^q$;
$-14a^m+15b^n-17c^p+19d^q$; $18a^m-b^n+c^p-d^q$;
$-13a^m+5b^n-3c^p+11d^q$.

7. $5a^2b-7a^3bc-13b^3c^4+10$; $\frac{1}{2}a^2b+8a^3bc-10b^3c^4+\frac{2}{15}$;
$-\frac{7}{8}a^2b-15a^3bc-7b^3c^4-\frac{4}{21}$;
$-1\frac{1}{2}a^2b+23a^3bc+\frac{1}{8}b^3c^4+1\frac{1}{14}$;
$-\frac{8}{10}a^2b-1\frac{3}{15}a^3bc+\frac{1}{5}b^3c^4+\frac{4}{6}$.

8. $7a-3b+5c-10d$; $2b-3c+d-4e$;
$5c-6a-4e+2d$; $-3b-8c+7a-e$;
$21e-16c+a-5d$.

9. $3ab^2-4a^2b+a^3$; $-4ac^2+5ab^2-c^3$;
$-7b^3+2a^2b-6ac^2$; $5a^3-11ab^2-12ac^2$.

10. $a^3+3a^2b+3ab^2+b^3$; $-5ab^2+3a^2b-b^3+3a^3$;
$+3ab^2-5a^2b+3b^3-3a^3$;
$-5b^3+2a^2b-4a^3+3ab^2$;
$7a^3+6b^3-5a^2b+5ab^2$; $4a^2b+3ab^2-3b^3+4a^3$.

11. $4x^4y^5z^6-3x^3y^4z^5+17x^3y^3z^4-8xy^2z^3$;
$14x^3y^3z^4+4xy^2z^3+5x^3y^4z^5-3x^4y^5z^6$;
$-x^4y^5z^6-2x^3y^4z^5+4xy^2z^3+19x^2y^3z^4$;
$2x^3y^4z^5+5xy^2z^3-7x^4y^5z^6+9x^2y^3z^4$;
$-12xy^2z^3+4x^4y^5z^6-15x^2y^3z^4-x^3y^4z^5$;
$3x^4y^5z^6+41x^2y^3z^4-x^3y^4z^5+7xy^2z^3$.

12. $a^m + 6a^{m-1}b + 10a^{m-2}b^2 + 6a^{m-3}b^3 + b^4$;
$-7b^4 - 3a^{m-2}b^2 + 7a^{m-3}b^3 + 8a^m + 4a^{m-1}b$;
$8a^{m-2}b^2 + 3a^{m-3}b^3 - 5a^m + 3b^4 + 6a^{m-1}b$;
$-3a^{m-1}b + 5a^{m-2}b^2 + 3a^m - 5b^4 - 3a^{m-3}b^3$.

13. $ax^3 + bx^2 - cx$; $5x^2 - bx^3 + 7x$; $-dx^3 + 4x - cx^2$;
$dx^2 + fx^3 - gx$; $mx^3 - nx^2 + px$; $-3x^3 + 7ax^2 - 3dx$.

14. $mx^4y^3 + nx^2y^2 - px^2y + 9x$; $-mx^3y^2 + 8x^4y^3 - px + px^2y$;
$-ax + px^2y - px^4y^3 + mx^3y^2$; $x^4y^3 - x^3y^2 + x - x^2y$;
$px + ax^2y - 2mx^3y^2 + 5mx^4y^3$.

15. $x^p + y^q + z^k - t^m$; $abx^p - mnz^k + amy^q + bt^m$;
$-5abx^p + 3aby^q + 8t^m - az^k$; $at^m - 3bz^k + mx^p + 14y^q$;
$3bcx^p - 4y^q + 3mz^k + nt^m$.

16. $abm^{p+2} + bcn^{q-1} - cdu^{y-3} + fgv^x$;
$-3a^2bm^{p+2} + 3bc^2n^{q-1} + 6c^2d^2u^{y-3} + 5fgv^x$;
$-cn^{q-1} + 3a^2bm^{p+2} - ku^{y-3} + av^x$;
$+5m^{p+2} - 3bn^{q-1} - 3c^2d^2u^{y-3} - 8fgv^x$;
$a^2m^{p+2} - bcn^{q-1} - fv^x + cdu^{y-3}$.

Ex. 3.

Subtraction.

1. From $m^2 - n^2$ take n^2.
2. From $m^2 - n^2$ take m^2.
3. From n^2 take $m^2 - n^2$.
4. From m^2 take $m^2 - n^2$.
5. From $-m^2$ take $m^2 + n^2$.
6. From $-x^n$ take $x^n - y^p$.
7. From $-y^p$ take $x^n - y^p$.
8. From $x^n - y^p$ take $-x^n$.
9. From $x^n + y^p$ take $-y^p$.
10. From y^p take $-x^n - y^p$.
11. From $m - n$ take $2m$.
12. From $m + n$ take $2n$.
13. From $2m$ take $m + n$.
14. From $3n$ take $m + n$.
15. From $8n$ take $3n - 5m$.
16. From $a - 5b$ take $+3b$.
17. From $a - 4b$ take $-3a$.
18. From $2a - b$ take $-2b$.
19. From $a + 2b$ take $+3b$.
20. From $3a - 4b$ take $-b$.

21. From $\frac{3}{8}n-\frac{1}{4}$ take $\frac{3}{8}n+\frac{1}{4}$.

22. From $\frac{4}{5}a-\frac{3}{7}b$
take $\frac{3}{7}b-\frac{4}{5}a$.

23. From $\frac{2}{3}x+\frac{4}{9}y$
take $\frac{2}{3}x-\frac{4}{9}y$.

24. From $1\frac{7}{8}m^5+\frac{5}{8}$
take $1\frac{7}{8}m^5-\frac{3}{8}$.

25. From $\frac{2}{3}a-\frac{4}{5}b$
take $\frac{5}{6}a-\frac{7}{8}b$.

26. From $\frac{3}{4}a-\frac{5}{6}b$ take $-\frac{3}{8}a$.

27. From $\frac{5}{6}x-\frac{7}{8}y$ take $\frac{2}{3}y$.

28. From $\frac{4}{5}x^3-\frac{2}{3}x^2$
take $-\frac{5}{7}x^2$.

29. From $1\frac{1}{2}a^2-1\frac{5}{4}b^2$
take $+2\frac{1}{2}b^2$.

30. From $a-b$ take $+\frac{5}{6}b$.

31. From $n+x$ take $x-1$.

32. From $n-x$ take $1-x$.

33. From $m-x$ take $m+1$.

34. From $x+y$ take $y-a$.

35. From $a+b$ take $a+c$.

36. From $a+2$ take $b-3$.

37. From $a-5$ take $5-b$.

38. From $2a-3b$ take $2b-3a$.

39. From $3x-y$ take $y+3x$.

40. From $7x-5y$ take $2y-x$.

41. From $4x^2+2xy+3y^2$ take $x^2-xy+2y^2$.

42. From $a^3+3a^2b+3ab^2+b^3$ take $a^3-3a^2b+3ab^2-b^3$.

43. From $a^4-4a^3b+6a^2b^2-4ab^3+b^4$
take $a^4+4a^3b+6a^2b^2+4ab^3+b^4$.

44. From $2x+11a+10b-5c-23$
take $2c-10+5a-3b$.

45. From $x^4+3ax^3-2bx^2+3cx-4d$
take $3x^4+ax^3-4bx^2-3cx+d$.

46. From $72x^4-78x^3y-10x^2y^2+17xy^3+3y^4$
take $-x^4+36x^3y-17xy^3-34y^4+10x^2y^2$.

47. From $6x^2+7xy-5y^2-12xyz-8yz-5z^2$
take $-8xy-7yz+3x^2-4y^2+5xyz$.

48. From $-a^2x^2+4axy-3aby^2$
take $4aby^2-5axy+2a^2x^2$.

49. From $10a^m-15b^n-c^p+5d^q$
take $-9a^m+2b^n+c^p-5d^q$.

50. From $\frac{5}{2}x^3 + 3ax - \frac{7}{8}a^2$ take $2x^2 - ax - \frac{1}{2}a^2$.

51. From $\frac{1}{2}x - 1\frac{3}{6}a + \frac{1}{3}b - \frac{1}{4}y$ take $-\frac{1}{2}y - \frac{3}{5}b + \frac{1}{6}a + 1\frac{1}{4}x$.

52. From $\frac{5}{2}a - 76 - 3bc + \frac{1}{2}c$ take $\frac{3}{4}a - 56\frac{4}{5} + \frac{1}{4}c - 3bc$.

53. From $\frac{1}{2}y - \frac{5}{2}a - \frac{3}{4}x + \frac{1}{3}b$ take $3y + 1\frac{1}{4}a - \frac{3}{8}x$.

54. From the sum of the first four following polynomials take the sum of the last four:
$m^2 + n^2 + p^2 + q^2$, $q^2 + p^2 + n^2$, $m^2 - p^2 + n^2 - q^2$,
$m^2 - n^2 + p^2 + q^2$, $n^2 + p^2 + q^2 - m^2$.

55. From $x^4 + 3ax^3 - 2bx^2 + 3cx - 4d$
take $3ax^4 + ax^3 - 4x^2 + 6cx + d$.

56. From $ax^3 + bx^2 + cx + d$ take $ex^3 - ax^2 + bx + 4$.

57. From $ax^3 + bc^2y - cxy^2 - dy^3$
take $ey^3 + fxy^2 - gx^2y - bx^3$.

58. From $5a^2xy^3 - 4bxy^3 + 9ax^3y + 15y^4$
take $3a^2xy^3 + 5b^2x^2y - 9cxy^3 - 5ay^4$.

59. From $mx^5 - 4mnx^4y + 3mx^3y^2 - 7mx^2y^3$
take $-x^5 + 7abx^3y^2 - 7px^4y + 7x^2y^3$.

60. From $abx^5 - bcx^4 + 3cdx^3 - 2dfx^2$
take $2fgx^2 - 6dex^3 - acx^4 + bcx^5$.

61. From $3mnyz + 7pqy^2z^2 + 4aby^3z^3$
take $-14qy^2z^2 + 9myz - 8ay^3z^3$.

62. From $48a^3b^2 + 56a^2b^3 - 34ab^4 + 59b^5$
take successively the following polynomials:
$13a^3b^2 - 14a^2b^3 + 26ab^4 - b^5$;
$15a^3b^2 + 70a^2b^3 - 60ab^4 + 30b^5$;
$20a^3b^2 + 30b^5$.

If $m = a^2 + b^2 + c^2$, $n = a^2 + b^2 - c^2$, $o = a^2 - b^2 + c^2$, $p = b^2 + c^2 - a^2$, find the value of:

63. $m + n + o + p$.

64. $m - n + o - p$.

65. $m - n - o + p$.

66. $m - n - o - p$.

If $x = 3a^2 - 2ab + 5b^2$, $y = 7a^2 - 8ab + 5b^2$,
$z = 9a^2 - 5ab + 3b^2$, $t = 11a^2 - 3ab - 4b^2$,
find the value of:

67. $x + y + z + t$.
70. $x + y - z - t$.
68. $x - y - z + t$.
71. $x - (y - z - t)$.
69. $y - (x + z - t)$.
72. $y - \{z - [x - (y + t)]\}$.

If $E = 5a^3 + 3a^2b - 7b^3$, $F = 8a^3 - 9a^2b - 3b^3$,
$G = 9a^2b - 3a^3 - 7b^3$, $H = 8a^2b - 7a^3 - 2b^3$,
find the value of:

73. $E - \{F - (G - H)\}$.
74. $G - \{F - [H - (E + G)]\}$.

Ex. 4.

Multiplication.

Multiply:

1. $2x - 4y + z$ by $3x$.
2. $a^2 + 2ab - b^2$ by a^2b^2.
3. $4a + 2ac^3 - 3cd^4 - 1$ by $-a^2d$.
4. $\frac{3}{2}b^2 - 4c^3 + \frac{2}{3}ad^2 - 3$ by $-\frac{3}{8}a^2c$.
5. $-5b^2 + 2bc^3 - 4c$ by $\frac{1}{2}b^2c^2$.
6. $x^3 - 3x^2y + 3xy^2 - 2$ by $-3xy^2$.
7. $3x^3 - 2xy + 4y^2$ by $2a^2x$.
8. $-2a^2b^2 + 5ab^3 - 7b^4$ by $-4ab$.
9. $a^2 - ax + x^2$ by $-ax$.
10. $x^3 - 3x^2y + 3xy^2 - y^3$ by xy.
11. $3x^{3m-1} - \frac{3}{7}y^{3n-5} + x^{2m}y^{3n} - y^3 - 3$ by $-x^{3-2m}y^{5-3n}$.
12. $-a^xb^{3y} + 3a^{4+x}b^{3+2y} - 5a^{3+x}b^{5+2y} + 2$ by $-\frac{5}{6}a^{1-x}b^{3-2y}$.
13. $-25c^p + 14c^{p+1}d + 40c^{2p+1}d^n$ by $-36c^{3-p}d^{3-2n}$.
14. $18m^{x+2}y^4 - 24m^{2x-4}y^5 + 1$ by $75m^{5-x}y^{p-4}$.

15. $34b^{m+n}d^{p-2q} - 27b^{2m-n-1}d^{p-q+2} + \frac{1}{2}b^{1-n}d^{q-1}$ by $-50b^{2n-2m+4}d^{2p+2q}$.

16. $39d^{x+y-1} - 54d^{x-2y+1} + 60d^{x+3y}$ by $30d^{2-x+2y}$.

17. $24x^{m+2n-1} - 42x^{2m-3n+2} + 25x^{2n+3m-2}$ by $25x^{2-m-4n}$.

18. $70y^{m-p-1} - 65y^{2-3m-2p} + 5y^{2p+m}$ by $\frac{3}{5}y^{2p-m+1}$.

19. $3a - 4b$ by $a - b$.

20. $4b - 5c$ by $3b + 4c$.

21. $4x + 9y$ by $x - 5y$.

22. $3a - 2b$ by $2a - b$.

23. $5a^2 - 3ab - 2b^2$ by $a^2 + 2ab$.

24. $8x^3 - 4x^2y + 2xy^2 - y^3$ by $2x - 3y$.

25. $3a^2 - 5ab + 2b^2$ by $a^2 - 7ab$.

26. $5a^3 - 2a^2x + ax^2$ by $2a^2 - ax + x^2$.

27. $3x^2 - 4x + 7$ by $5x^2 - x - 4$.

28. $x^2 + 7x - 5$ by $x^2 - 3x + 7$.

29. $3a^3b - 2a^2b^2 + ab^3$ by $2a^2 - ab - 5b^2$.

30. $x^3 - 6ax^2 + 12a^2x - 8a^3$ by $x^2 - 4ax + 4a^2$.

31. $x^3 - 2x + 1$ by $x^2 - 3x + 2$.

32. $x^4 - 7x^2y^2 + 6xy^3 - y^4$ by $x^2 - 2xy^2 + y^3$.

33. $x^2 - 5ax - 2a^2$ by $x^2 + 2ax + 3a^2$.

34. $2x^4 - 4x^2 - 4x - 1$ by $2x^4 - 4x^2 - 4x - 1$.

35. $x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4$ by $x^2 - 3x^2y + 3xy^2 - y^3$.

36. Expand $(a - 2b + 3c)^3$.

Multiply:

37. $3x^4 + 6x^3y - 9xy^3 + 12y^4$ by $2x^2 - 3xy + y^2$.

38. $3a^3x - 2a^2x^2 + 5ax^3 - 8x^4$ by $-2a^2 + 6ax - 4x^2$.

39. $2by^5 - 3b^2y^4 + 4b^3y^3 - 5b^4y^2$ by $y^3 - 4by^2 + 6b^2y$.

40. $c^3 - 8c^2z + 5cz^2 + 11z^3$ by $6c^2 - 2cz + 12z^2$.

41. $a^4 - 7a^3b + 6a^2b^2 + 8ab^3 - 2b^4$ by $a^2 - 3ab + 2b^2$.

42. $x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4$ by $x^2 + 2xy + y^2$.

43. $3a^4b^2 - 6a^3b^3 + 4a^2b^4 + ab^5$ by $2a^2b - 5ab^2 + 2b^3$.

44. $x^4 - 3x^3 + 2x^2 - x + 5$ by $3x^3 - 2x^2 - 5x + 3$.

45. $7a - 2a^2 + 8a^3 - 4a^4$ by $5a + 3a^2 - 7a^3$.

46. $2p^3 + 4p^2q - 3pq^2 - q^3$ by $5p^2 - 2pq - q^2$.

47. $a^5 - 2a^4b + 3a^3b^2 - 2a^2b^3 + ab^4 - b^5$ by $2a^2 - ab + b^2$.

48. $5c^8 - 3c^6 + 9c^4 - 10c^2 + 4$ by $9c^6 + 5c^4 - 13c^2$.

Expand:

49. $(a^2 - 5ab - b^2)^3$.

50. $(x^3 - 3x^2 - 2x)^4$.

Multiply:

51. $2d^5 - 7d^4e + 3d^3e^2 - 9d^2e^3 + 6de^4 + 8e^5$
by $7d^3 + 24d^2e + 70de^2 - 2e^3$.

52. $a^3 - 2a^2x + ax^2 - \frac{1}{3}x^3$ by $3a^2 - ax + 3x^2$.

53. $c^4 - 2c^3 - \frac{1}{2}c^2 + 120c - 8$ by $50c^3 + 18c^2 - \frac{1}{10}c - \frac{1}{2}$.

54. $d^5 - 5cd^4 + \frac{1}{3}c^2d^3 - 9c^3d^2 + \frac{1}{4}c^4d$
by $8d^3 + 7cd^2 - \frac{1}{3}c^3d - c^3$.

55. $24p^5 - 36p^4q - 16p^3q^2 + 20p^2q^3 - 16pq^4 + \frac{1}{2}q^5$
by $-\frac{1}{4}p^3 + 8p^2q + 7pq^2 - 11q^3$.

56. $14c^4 - 18c^3d + 15c^2d^2 - 8cd^3 + \frac{1}{3}d^4$
by $12c^5 + \frac{1}{3}c^4d - 10c^3d^2 - \frac{1}{2}c^2d^3$.

Expand:

57. $\left(\frac{c^2}{3} - \frac{c}{4} - \frac{1}{2}\right)^3$.

58. $\left(\frac{x}{2} - \frac{x^2}{3} + \frac{x^3}{4}\right)^2$.

Multiply:

59. $0.5m^4 - 0.4m^3n + 1.2m^2n^2 + 0.8mn^3 - 1.4n^4$
by $0.4m^2 - 0.6mn - 0.8n^2$.

60. $7x^2 + y^2 - 3xy$ by $2x^2 + y - x$.

61. $2x - x^2 + 4x^3 - 10 + 7x^5$ by $2x^4 - 2x^2 + 3x^3 + 2 + 4x$.

62. $2x^4 - 3x^3y + 2y^4 - 2xy^3 + 4x^2y^2$ by $2xy^2 - y^3 + \frac{1}{2}x^3 - \frac{1}{3}x^2y$.

63. $2b^3d^2+3b^4-7bd^3-5b^3d+d^4$ by $3b^2+3d^2-5bd$.

64. Expand $(-7ab^2+2a^3-3a^2b)^3$.

Multiply:

65. $5p^2x^2+3p^4-9px^3-3p^3x$ by $2p^2x-8x^3+9p^3-7px^2$.

66. $4b^3-2b+1-3b^2$ by $4b^2+5b+2b^3$.

67. $a^3+3a-2a^2+1$ by $4a^2+3-2a$.

68. $bc+ac+ab$ by $a-b+c$.

69. $1-2x+x^2$ by $1-2y+y^2$.

70. $bxz-cxy+ayz$ by $bcx+acy-abz$.

71. x^2+bx-c by x^2-ax+b.

72. x^2-ax-b^2 by $px+x^2-q^2$.

73. $x^2-3ax-6a^2$ by $5bx+8b^2+x^2$.

74. $bx-ax^2+x^3-c$ by $n-mx+x^2$.

75. $bx-ax^2+x^3-c$ by $n+mx+x^2$.

76. $1-ax+bx^2-cx^3$ by $1-x^2+x$.

77. az^2+bz+c by dz^2+ez+f.

78. mz^3-nz^2+q-pz by sz^2+rz^3-u+pz.

79. a^3+3a^2-4a by $-4na+3ma^2+p$.

80. $5nx^3+4mx^3-2px-v$ by $2ax^2-3bx+4d$.

81. $a^m+2a^{m+1}-3a^{m+2}-1$ by $a+1$.

82. $a^n-4a^{n-1}+5a^{n-2}+a^{n-3}$ by $a-1$.

83. $a^n-a^{n-1}b+a^{n-2}b^n$ by $a-b$.

84. $x^{n-1}-x^{n-2}y+x^{n-3}y^2-x^{n-4}y^3+x^{n-5}y^4$ by $x+y$.

85. $a^p-3a^{p-1}+4a^{p-2}-6a^{p-3}+5a^{p-4}$ by $2a^3-a^2+a$.

86. $a^{2n+1}-a^{n+1}-a^n+a^{n-1}$ by $a^{n+2}-a^2-a+1$.

87. $3x^{4n+1}-4x^{3n}+2x^{2n-1}-x^{n-2}$
by $2x^{4n+1}-5x^{3n}-2x^{2n-1}+x^{n-2}$.

88. $m^p+3m^{p-1}n+m^{p-2}n^2-m^{p-3}n^3$ by $m^r-3m^{r-1}n-3m^{r-2}n^2$.

89. Expand $(e^m-e^{m-1}d+e^{m-2}d^2-e^{m-3}d^3)^2$.

Multiply:

90. $2x^{r+5}y^{q-1}+3x^{r+4}y^{2q-1}-x^{r+3}y^{3q-1}+4x^{r+2}y^{4q-1}$
by $2x^{r+3}y^{3-4q}-3x^{r+1}y^{3-3q}+x^{r}y^{3-2q}+4x^{r-1}y^{3-q}$.

91. $a^{p+4}b^{1-q}+a^{p+3}b^{2-q}-2a^{p+2}b^{3-q}-4a^{p+1}b^{4-q}+a^{p}b^{5-q}$
by $2a^{5-p}b^{q}-2a^{4-p}b^{q+1}+6a^{3-p}b^{q+2}-2a^{2-p}b^{q+3}$
$+4a^{1-p}b^{q+4}$.

92. $u^{p+1}-3u^{p+2}+u^{p+3}-2u^{p+4}$ by $2u^{q-p}+3u^{2q-p}-4u^{3q-p}$.

93. $5x^{p-3}y^{r+3}-2x^{p-1}y^{r+1}-x^{p-2}y^{r+2}$
by $3x^{p+4}y^{r-1}+4x^{p+5}y^{r-2}-x^{p+3}y^{r}$.

94. $a^{p}+3a^{p-2}-2a^{p-1}$ by $2a^{p+1}+a^{p+2}-3a^{p}$.

95. $3y^{m}-2y^{m+1}-5y^{m+2}+y^{m+3}$
by $3y^{p-3}+2y^{p-2}-5y^{p-1}-y^{p}$.

Ex. 5.

Division.

Divide:

1. $5a^{6}+15a^{5}+5a+15$ by $a+3$.
2. $35x^{3}+47x^{2}+13x+1$ by $5x+1$.
3. $z^{4}-bz^{3}-b^{2}z^{2}-b^{3}z-2b^{4}$ by $z-2b$.
4. $2a^{4}-6a^{3}+3a^{2}-3a+1$ by $a^{2}-3a+1$.
5. $42a^{4}+41a^{3}-9a^{2}-9a-1$ by $7a^{2}+8a+1$.
6. $x^{6}+4x^{5}-3x^{4}-16x^{3}+11x^{2}+12x-9$
by $x^{3}+x^{2}-5x+3$.
7. $6b^{5}+20b^{4}-12b^{3}-48b^{2}+22b+12$ by $6b^{2}+14b+4$.
8. $24m^{5}+16m^{4}n-36m^{3}n^{2}-36m^{2}n^{3}-8mn^{4}$
by $2m^{2}-2mn-n^{2}$.
9. $d^{6}-3d^{5}+6d^{4}-7d^{3}+6d^{2}-3d+1$ by $d^{2}-d+1$.
10. $2m^{4}-6m^{3}+3m^{2}-3m+1$ by $m^{2}-3m+1$.
11. $p^{6}-p^{5}+2p^{4}-p^{3}+2p^{2}-p+1$ by $p^{4}-2p^{3}+3p^{2}-2p+1$.
12. $6a^{3}x-17a^{2}x^{2}+14ax^{3}-3x^{4}$ by $2a-3x$.

13. $y^5 - 2y^4 - 6y^3 + 4y^2 + 13y + 6$ by $y^3 + 3y^2 + 3y + 1$.

14. $4y^4 - 18y^3 + 22y^2 - 7y + 5$ by $2y - 5$.

15. $4x^2 - 28xy + 49y^2$ by $2x - 7y$.

16. $4a^3 + 4a^2 - 29a + 21$ by $2a - 3$.

17. $6a^3 - 13a^2b + 4ab^2 + 3b^3$ by $2a - 3b$.

18. $6x^4 - 13ax^3 + 13a^2x^2 - 13a^3x - 5a^4$ by $2x^2 - 3ax - a^2$.

19. $a^5 + a^2x^3 + a^4x + a^3x^2 + ax^4 + x^5$ by $a + x$.

20. $45x^4 + 35x^2 - 4 + 18x^3 + 4x$ by $9x^3 + 7x - 2$.

21. $6x^4 + 13xy^3 - x^3y + 4y^4 + 2x^2y^2$ by $2x^2 - 3xy + 4y^2$.

22. $x^6 - a^4x^2 - a^2x^4 - 2ax^5 - 2a^5x + 4a^3x^3 + a^6$
by $x^4 + 2xa^3 - 2x^3a - a^4$.

23. $2x^3 + 7x^2y - 9y^3(x + y)$ by $2x - 3y$.

24. $2x^4 + x^3y - 3xy^3 - 13x^2y^2 + y^4$ by $x^2 - 2xy - y^2$.

25. $x^5 + 4x^3 - 92x - 3x^4 + 26x^2 + 55$ by $x^2 - 3x + 11$.

26. $x^7 - 3x^6 + x^5 - 4 + 12x - 4x^3$ by $x^5 - 4$.

27. $15x^5 - 3xy^4 + 4x^3y^2 + 10x^4y + 6x^2y^3$ by $5x^3 + 3xy^2$.

28. $x^5 - 125x^2 - 9x^3 + 5x^4 - 340x - 300$ by $x^3 - 15x - 50$.

29. $5x^7 - 4x^5y^4 - 22x^6y - 6x^4y^3 + 12x^5y^2 + 8x^2y^5$
by $x^3 - 4x^2y + 2y^3$.

30. $21a^4 - 16a^3b - 5ab^3 + 16a^2b^2 + 2b^4$ by $3a^2 - ab + b^2$.

31. $\frac{2}{3}b^3 + 1\frac{3}{8}bd^2 - \frac{4}{2}\frac{3}{7}\frac{3}{0}b^2d - \frac{5}{12}d^3$ by $\frac{3}{2}b - \frac{5}{8}d$.

32. $-\frac{1}{8}d^5 + d^2 - \frac{4}{2}\frac{1}{4}d^3 + \frac{5}{6}d^4$ by $-\frac{3}{4}d^2 + 2d$.

33. $\frac{9}{16}a^4 - \frac{7}{8}a^3b + \frac{1}{3}\frac{9}{6}a^2b^2 + \frac{1}{6}ab^3$ by $\frac{3}{2}a + \frac{1}{3}b$.

34. $\frac{1}{14}m^3 + \frac{1}{24}m^2n - \frac{2}{1}\frac{5}{2}mn^2 + \frac{1}{9}\frac{4}{6}n^3$ by $\frac{1}{2}m - \frac{7}{8}n$.

35. $\frac{2}{15}x^3 - \frac{1}{12}x^2y - \frac{1}{2}\frac{9}{4}xy^2 + \frac{5}{12}y^3$ by $\frac{1}{3}x - \frac{5}{6}y$.

36. $\frac{1}{2}a^5 + \frac{2}{2}\frac{3}{4}a^4b - \frac{5}{7}\frac{9}{2}a^3b^2 + 1\frac{3}{4}a^2b^3 - 1\frac{2}{9}ab^4 + \frac{2}{9}b^5$
by $\frac{3}{4}a^2 + 2ab - \frac{2}{3}b^2$.

37. $x^5 - \frac{2}{3}\frac{3}{0}x^4 + \frac{3}{1}\frac{1}{0}x^3 - \frac{7}{3}x^2 - \frac{1}{1}\frac{8}{8}\frac{1}{1}x + \frac{5}{3}$ by $x^2 - \frac{x}{6} + 5$.

38. $\frac{1}{3}a^5 + 1\frac{1}{4}\frac{1}{5}a^4b - 2\frac{1}{8}\frac{8}{0}a^3b^2 - 2\frac{1}{1}\frac{1}{5}a^2b^3 + \frac{7}{1}\frac{1}{0}ab^4 - 3b^5$
by $\frac{2}{3}a^2 - \frac{2}{5}ab - 4b^2$.

39. $\frac{16}{15}y^7+\frac{2}{15}y^6z-3\frac{1}{80}y^5z^2+\frac{283}{180}y^4z^3+1\frac{2}{35}y^3z^4-\frac{7}{12}y^2z^5-\frac{5}{8}yz^6$ by $\frac{8}{5}y^3+\frac{2}{3}y^2z-\frac{5}{8}yz^2$.

40. $2z^7+\frac{14}{15}z^6-2\frac{11}{30}z^5+11\frac{47}{150}z^4-13\frac{1}{3}z^3+1\frac{41}{63}z^2-\frac{6}{7}z+\frac{4}{7}$ by $\frac{4}{5}z^2-\frac{2}{3}z+2$.

41. $-3a^7-\frac{11}{40}a^6b+\frac{267}{720}a^5b^2+\frac{419}{120}a^4b^3-\frac{17}{72}a^3b^4+\frac{1}{6}a^2b^5+8\frac{1}{2}ab^6-6b^7$ by $1\frac{1}{5}a^4-\frac{1}{4}a^3b-\frac{2}{3}a^2b^2+1\frac{1}{2}ab^3-b^4$.

42. x^3-8a^2+2ax by $2a-x$.

43. $-73x^3-25+56x^4+95x-59x^5$ by $-11x+7x^3-3x^2+5$.

44. $26x^3y^3+x^6+6y^6-17xy^5-5x^5y-2x^4y^2-x^2y^4$ by x^2-3y^2-2xy.

45. $48x^2y^3+26xy^4-17x^3y^2+8x^5-22x^4y-8y^5$ by $2x^2-4y^2-3xy$.

46. $8ab^6-8a^6b+24a^2b^5+a^5b^2+14a^3b^4+12a^4b^3+5a^7$ by $7a^2b+2ab^2+5a^3$.

47. $62x^5y^3+43x^7y^5-55x^6y^4-55x^4y^3-6x^3-21x^5y^6+28x^3y$ by $4x^3y-2x-3x^3y^3$.

48. $4y^3z^6+4y^3z-38y^7z^5-42y^5z^3+51y^6z^4+52y^4z^3-15y^3$ by $-3y+4y^4z^3-2y^3z^2+5y^2z$.

49. $1\frac{1}{6}x^5+\frac{11}{144}a^4x+\frac{299}{480}a^2x^3-\frac{7}{8}ax^4-1\frac{1}{2}a^3x^2+\frac{5}{9}a^5$ by $\frac{5}{8}ax-1\frac{1}{2}x^2+\frac{2}{3}a^2$.

50. $7a^4b^3-19a^2b^5+a^7-9a^5b^2+13a^3b^4-b^7+8ab^6$ by $b^3-2ab^2+a^3$.

51. $2a^3-16a+6$ by $a+3$.

52. $49x^3-72xy^2+27y^3$ by $7x-3y$.

53. $4x^5y-4x^3y^3+4x^2y^4-xy^5$ by $2x^2-2xy+y^3$.

54. $4x^4-x^2y^2+6xy^3-9y^4$ by $2x^2-xy+3y^2$.

55. $x^6-5x^4+3x^3+6x^2-7x+2$ by x^3-3x^2+3x-1.

56. x^3y^3-64 by $xy-2$.

57. $y^{10}+y^9+y^8+2y^7+2y^4+2y^3+y^2+y+1$ by $y+1$.

58. a^4-b^4 by $a^3+a^2b+ab^2+b^3$.

59. $3x^5+4x^3y^2+4x^2y^3+xy^4+4y^5$ by $x^3+x^2y+xy^2+y^3$.

60. $12a^4-27a^2b^3+18ab^3-3b^4$ by $6a^2-9ab+3b^2$.

61. $a^5-5a^3b^2-5a^2b^3+b^5$ by $a^2-3ab+b^2$.

62. $a^4+4ab^3-4abc+3b^4+2b^2c-c^2$ by $a^2-2ab+3b^2-c$.

63. $4x^4-5x^2y^2-8x^2+y^4-4y^2+4$ by $2x^2-3xy+y^2-2$.

64. $a-4a^3+12a^4-9a^5$ by $1+2a-3a^2$.

65. $a^{m+2}-a^{m+3}+37a^{m+5}-55a^{m+6}+50a^{m+7}$ by $a^2-3a^3+10a^4$.

66. $6d^{x+3}-5d^{x+2}-6d^{x+1}+19d^x-21d^{x-1}+4d^{x-2}$
by $2d^3+d^2-4d$.

67. $4h^{x+1}-30h^x+19h^{x-1}+5h^{x-2}+9h^{x-4}$
by $h^{x-3}-7h^{x-4}+2h^{x-5}-3h^{x-6}$.

68. $6x^{m-n+2}+x^{m-n+1}-22x^{m-n}+19x^{m-n-1}-4x^{m-n-2}$
by $3x^{3-n}-4x^{2-n}+x^{1-n}$.

69. $6b^{x+y+2}+b^{x+y+1}-9b^{x+y}+11b^{x+y-1}-6b^{x+y-2}+b^{x+y-3}$
by $2b^{y+2}+3b^{y+1}-b^y$.

70. $6x^{3n+2}-23x^{7n+1}+18x^{6n}-x^{5n-1}-3x^{4n-2}+4x^{3n-3}-x^{2n-4}$
by $2x^{4n+1}-5x^{3n}-2x^{2n-1}+x^{n-2}$.

71. $a^{m+n}b^n-4a^{m+n-1}b^{2n}-27a^{m+n-2}b^{3n}+42a^{m+n-3}b^{4n}$
by $a^nb^n-7a^{n-1}b^{2n}$.

72. $x^{mn}-x^ny^{(n-1)m}-y^mx^{(m-1)n}+y^{mn}$ by x^n-y^m.

73. $x^{3n}-3x^{2n}y^n+3x^ny^{2n}-y^{3n}$ by x^n-y^n.

74. $a^{m+n}b^n-4a^{m+n-1}b^{2n}-27a^{m+n-2}b^{3n}+42a^{m+n-3}b^{4n}$
by $a^m+3a^{m-1}b^n-6a^{m-2}b^{2n}$.

Perform the following divisions, and continue the operation to the fifth term of the quotient:

75. $\frac{1}{1+x}$.

76. $\frac{1}{1-x}$.

77. $\frac{x}{x+1}$.

78. $\frac{x}{x^2-1}$.

79. $\frac{a}{1+x}$.

80. $\frac{m}{1-y}$.

81. $\frac{1+2x}{1-3x}$.

82. $\frac{1-2a}{1+2a}$.

83. $\frac{x+y}{x-y}$. 85. $\frac{ay-1}{by-1}$. 87. $\frac{1}{x-2}$. 89. $\frac{1}{-2+x}$.

84. $\frac{1-ax}{1+bx}$. 86. $\frac{1}{1-2x+x^2}$. 88. $\frac{1}{3+x}$. 90. $\frac{1+x}{1-2x+x^2}$.

Calculate to five decimals the numerical values of

91. $1+\frac{x}{1}+\frac{x^2}{1\times 2}+\frac{x^3}{1\times 2\times 3}+\frac{x^4}{1\times 2\times 3\times 4}$ ..... for $x=1$. (1)

92. $4(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}\ldots\ldots)$. (2)

93. $\frac{a}{1}-\frac{a^3}{3}+\frac{a^5}{5}-\frac{a^7}{7}+\frac{a^9}{9}$ ..... for $a=0.2$.

94. $x-\frac{x^3}{1\times 2\times 3}+\frac{x^5}{1\times 2\times 3\ldots\ldots 5}-\frac{x^7}{1\times 2\times 3\ldots\ldots 7}+\frac{x^9}{1\times 2\times 3\ldots\ldots 9}\ldots\ldots$
for $x=1.5708$ (6 terms). (3)

95. $2\left(\frac{1}{2n+1}+\frac{1}{3(2n+1)^3}+\frac{1}{5(2n+1)^5}+\frac{1}{7(2n+1)^7}+\ldots\ldots\right)$ (4)
for $n=1$.

(1) This series is the development of e^x; and, if $x=1$, it gives the value of e, the base of the system of Napierian logarithms.

(2) This series, known by the name of the Leibnitz series, gives the value of π; but a great number of terms must be calculated to obtain a near approximation.

(3) This series gives the value of the sine of an angle in the function of the arc x, reduced to the radius.

(4) This series, if $n=1$, gives the Napierian logarithm of 2.

Ex. 6.

Parentheses.

Simplify:

1. $(7a-2b)-[(3a-c)-(2b-3c)]$.
2. $(9a-4c)-\{[(3b-4c)+5a]-3b\}$.
3. $2a-(3b+2c)-\{5b-3a-(a+b)+5c-[2a-(c+2b)]\}$.

4. $a-\{2b+[3x-3a-(a+b)]+[2a-(b+c)]\}$.

5. $16-x-\{7x-[8-9x-(3-6x)]\}$.

6. $2a-\{3b+(2b-c)-4c+[2a-(3b-(c-2b))]\}$.

7. $x^4-\{4x^3-[6x^2-(4x-1)]\}-(x^4+4x^3+6x^2+4x+1)$.

8. $ab-[(3bce-2ab)-(5bce-bef)+(3ab-3bef)]$.

9. $1-[1-(2-x)]+[4x-(3-6x)]+[4-(6x-5)]$.

10. $4x-3y-\{(2x+4y)+3x+[y-9x-(2y-x)+(x-y)]\}$.

11. $11x-\{7x-[8x-(9x-(12a-6x))]\}$.

12. $[(2a-3)b^2+(3a-4)b-(2a-5)]\times[(a+1)b-(1-2a)]$.

13. $[(a^2+ab+b^2)m^3-(a+b)m^2-abm]\times[(a-b)m^3+2m-1]$.

14. $[2x^3-(a+b)x^2+abx-(a-b)]\times[(a-b)x^2+abx+(a+b)]$.

15. $[x^4-(a-1)x^3y+(a-2)x^2y^2-(a+2)xy^3]\times[x^2+(a+1)xy+(a-2)y^2]$.

16. $[(b-a)m^3+(a-b)m^2-2(a-b)m]\times[(a+b)m^2-2(b-a)m-2a]$.

17. $[(a+2b+1)x^2-(3a-2b)x+(2a-1)]\times[(a-2b+1)x^2+(3a+2b)x-2b]$.

18. $[(b^2-1)x^2-(b-2)x+(1-b)]^2$.

19. $[9a^2-(1-m)a-(1-m^2)]^3$.

20. $[(2a-3c)y^2-(a-c)y+(2a+c)]\times[(2d+3c)y^2+(a+c)y-(2a-c)]$.

21. $[x^3+(a+1)x^2-(a^2-2a-3)x+(a^3-5a^2+8a-7)]\times[x^2+(a-1)x+(a^2-3a+1)]$.

22. $[y^3+(a+b)y^2+(a^2-b^2)y+(a^3-3a^2b+3ab^2-b^3)]\times[y^2-(a-b)y+(a^2-2ab+b^2)]$.

23. $(x+y+z)(x+y-z)-(x-y+z)(-x+y+z)$.

24. $(15a^2-12b)(2a+3b)-(4a^2+15b)(5a^2-4b)$.

25. $(9a^2-4b^2)(a-1)-(3a-2b)(3a^2+ab-3a-2b)$.

26. $(5x-2y)(x^2-2xy+3y^2-8)+(2x^2+2xy-5y^2+10)$
$(5x-2y)-(3x^2-2y^2+2)(5x-2y)$.

27. $(a^2-b^2)(2b-3a)+(a+b)(8b-7a)a$
$-(a+b)(8b-7a)b$.

28. $15x^2+24y^2-(3x+2y)(5x+6y)$.

29. $26ab-(9a-8b)(5a+2b)-(4b-3a)(15a+4b)$.

30. $12ab+5b^2-(3a-4b)(2a+8b)-(3a-2)(a-b)$.

31. $(4x-3y)(7x+8y)-(8x-9y)(5x+7y)$
$-(3x-2y)(5x+8y)$.

32. $(a^2-b^2)(2a-3b+5c)+(b-a)(3a^2+4bc-5ac)$
$+(b^2-a^2)(4a-3b+c)$.

33. $(a^2-2ab+b^2)(3a^3-5a^2b+3b^3+a)$
$-(a^3-a^2b-ab^2+b^3)(5a^2-5ab+4b^2+1)$
$+(a-b)^2(2a^3+15a^2b+b^3)$.

34. $(x^3+3x^2y+3xy^2+y^3)(a^3-2a^2b+3b^4+1)$
$+(x+y)^3(2a^3+2a^2b-3b^4)-(x+y)^2(3a^3x+3a^2y-2a^2x)$.

35. $[2a^3+(2x-9)a^2-(9x+6)a+27]\div(2a-9)$.

36. $[a^4-pa^3+(q-1)a^2+pa-q]\div(a^2-1)$.

37. $[z^4-2a(a-b)z^3+(a^2+b^2)(a-b)z-a^2b^2]$
$\div[z^2-(a-b)z+b^2]$.

38. $[(a+2b)^2-(c-3d)^2]\div(a+2b+c-3d)$.

39. $(x^3+3xy+y^3-1)\div[x+(y-1)]$.

40. $[a(a-1)x^3+(a^3+2a-2)x^2+(3a^2-a^3)x-a^4]$
$\div(ax^2+2x-a^2)$.

41. $[(x^3-1)y^3-(x^3+x^2-2)y^2+(4x^2+3x+2)y$
$-3(x+1)]\div[(x-1)y^2-(x-1)y+3]$.

42. $(x^4-2x^3y+x^2y^2-a^2x^2+2a^2xy-a^2y^2)\div(x-a)$.

43. $[z^4-(m+n)^2z^2+2mn(m+n)z-m^2n^2]$.
$\div[z^2-(m+n)z+mn]$.

44. $[my^6+(m+n)y^5+(m+n+o)y^4+(m+n+o+p)y^3$
$+(n+o+p)y^2+(o+p)y+p]\div(y^3+y^2+y+1)$.

45. $[20(a+b)^5-46(a+b)^4x+84(a+b)^3x^2-78(a+b)^2x^3$
$+64(a+b)x^4-32x^5]\div[5(a+b)^3-4(a+b)^2x$
$+5(a+b)x^2-4x^3]$.

46. $[12x^5-31x^4y+(20y^3-29z^3)x^3+(37yz^2+18z^3)x^2$
$+(15z^4-24yz^3)x-30z^5]\div(3x^2-4xy-5z^2)$.

Ex. 7.

Multiplication and Division by Inspection.

$(a+b)^2=a^2+2ab+b^2$.
$(a-b)^2=a^2-2ab+b^2$.
$(a+b)^3=a^3+3a^2b+3ab^2+b^3$.
$(a-b)^3=a^3-3a^2b+3ab^2-b^3$.
$(a+b)(a-b)=a^2-b^2$.
$(x+a)(x+b)=x^2+(a+b)x+ab$.
$(x+a)(x-b)=x^2+(a-b)x-ab$.
$(x-a)(x-b)=x^2-(a+b)x+ab$.
$(a-b)(a^2+ab+b^2)=a^3-b^3$.
$(a+b)(a^2-ab+b^2)=a^3+b^3$.
$(a+b+c+d)^2=a^2+b^2+c^2+d^2+2ab+2ac+2ad+2bc+2bd+2cd$.
x^n-y^n is divisible by $x-y$ if n is a positive integer.
x^n-y^n is divisible by $x+y$ if n is an *even* positive integer.
x^n+y^n is divisible by $x+y$ if n is an *odd* positive integer.
x^n+y^n is not divisible by $x-y$.

Write by inspection:

1. $(x+y)^2$.
2. $(b-c)^2$.
3. $(m+n)^3$.
4. $(a-b)^3$.
5. $(m+2n)^2$.
6. $(2x-a)^2$.
7. $(3a-b)^3$.
8. $(4a-5b)^2$.
9. $(3x+5y)^2$.
10. $(8z-5)^2$.
11. $(3+2d)^3$.
12. $(-4m+5n)^2$.

13. $(1+2x^2)^2$.
14. $(3+5m^5)^2$.
15. $(4a-3a^2)^2$.
16. $(2x^2+5)^2$.
17. $(7c-4d)^2$.
18. $(8m+4n)^2$.
19. $(9p-3q)^2$.
20. $(5a^3-2b^4)^2$.
21. $(a^3-2b)^3$.
22. $(y+2z)^3$.
23. $(2u+v)^3$.
24. $(5a-b)^3$.
25. $(x^3+y^3)^3$.
26. $(9m^3-5n)^3$.
27. $(bc^3-ab^2)^3$.
28. $(m^2n+p^2q)^3$.
29. $(10-x^2)^3$.
30. $(6+5a^3)^3$.
31. $(7d^2-2)^3$.
32. $(8z^4-9)^3$.
33. $(m+n)(m-n)$.
34. $(1+x)(1-x)$.
35. $(3-y)(3+y)$.
36. $(3x-2y)(3x+2y)$.
37. $(3dy-1)(3dy+1)$.
38. $(5x^3-2y^3)(5x^3+2y^3)$.
39. $(3ab^2-5a^2b)(3ab^2+5a^2b)$.
40. $(4x^2y-6xy^2)(4x^2y+6xy^2)$.
41. $(a^2+3x)[-(3x-a^2)]$.
42. $(5-bx^3)(bx^3+5)$.
43. $(6m+7n^4)(7n^4-6m)$.
44. $(12-ab)(ab+12)$.
45. $(a^4x+ax^4)(ax^4-a^4x)$.
46. $(a+b-c)(a+b+c)$.
47. $(a-b+c)(a-b-c)$.
48. $(x^2+y^2-xy)(x^2+y^2+xy)$.
49. $(2x-y-3z)(2x-y+3z)$.
50. $(x-2y+3z)(x+2y+3z)$.
51. $(a+2b+3c+d)(a-2b+3c-d)$.
52. $(1+x-3x^3-2x^2)(1+x+2x^2+3x^3)$.
53. $(2+a^2+3a^3+d^2)(2-a^2+3a^3-d^2)$.
54. $(a-x)(a+x)(a^2+x^2)$.
55. $(m+1)(m+2)$.
56. $(b^2+2)(b^2+4)$.
57. $(x+2)(x+3)$.
58. $(3y^2+8)(3y^2+12)$.
59. $(x-2)(x+5)$.

60. $(3a-7)(3a+4)$.
61. $(2a-11)(2a+5)$.
62. $(4x-3)(4x-9)$.
63. $(5d-6)(5d-11)$.
64. $(m+2)(m+n)$.
65. $(1+3a)(1+b)$.
66. $(4a+1)(4a+b)$.
67. $(3a+4)(3a-n)$.
68. $(5x-a)(5x+1)$.
69. $(3x^2-m)(3x^2+p)$.
70. $(5y^4+3x)(5y^4+2a)$.
71. $(m^3+2)(m^3-2x)$.
72. $(x^2-xy+y^2)(x+y)$.
73. $(y^2-3yz+9z^2)(y+3z)$.
74. $(m^2+mn+n^2)(m-n)$.
75. $(4+2a+a^2)(2-a)$.
76. $(d^2-5d+25)(d+5)$.
77. $(a^6+3a^3+9)(a^3-3)$.
78. $(16m^4+4m^2y^4+y^8)(4m^2-y^4)$.
79. $(49x^6+56x^3y+64y^2)(7x^3-8y)$.
80. $(9m^2+3am+a^2)(3m-a)$.
81. $(2a+4b-5c+2d)^2$.
82. $(3x-2y+4z-5)^2$.
83. $(2p^2+3x^4-2xy-y^2)^2$.
84. $(a+2b^2+3c^3+4d^4)^2$.
85. $(a^2-5b^3+2a-3b^2)^2$.
86. $(\frac{1}{2}x^2-4y+\frac{2}{3}y^2+6z^3)^2$.
87. $(\frac{2}{5}a-\frac{3}{2}a^2-8b^2+\frac{1}{2}bc)^2$.
88. $(0.6m^2-\frac{1}{2}n+0.8p^3-3x^5)^2$.
89. $(0.3ax+x^2-2a^2+4x)^2$.
90. $(x^3-ax^2+2a^2x-a^3)^2$.
91. $(m^3+3dm^2+3d^2m+d^3)^2$.
92. $(y^4-2xy^3+3x^2y^2-4x^3y)^2$.
93. $(x^3+y^3)\div(x+y)$.
94. $(x^3-y^3)\div(x-y)$.
95. $(a^2-y^2)\div(a+y)$.
96. $(h^2-l^2)\div(h-l)$.
97. $(1+x^3)\div(1+x)$.
98. $(y^3-1)\div(y-1)$.
99. $(u^2-v^2)\div(u+v)$.
100. $(x^2-y^2)\div(x-y)$.
101. $(a^4-m^4)\div(a-m)$.
102. $(a^4b^4-x^4y^4)\div(ab-xy)$.
103. $(h^6-m^6)\div(h-m)$.
104. $(x^7-y^7)\div(x-y)$.
105. $(m^8-n^8)\div(m+n)$.
106. $(a^9+b^9)\div(a+b)$.

107. $(4h^2-m^2)\div(2h-m)$.

108. $(1+x^7)\div(1+x)$.

109. $(x^3-1)\div(x-1)$.

110. $(b^5x^5+c^5y^5)\div(bx+cy)$.

111. $(16-x^4)\div(2+x)$.

112. $(81-y^4)\div(3-y)$.

113. $(32a^5+b^5)\div(2a+b)$.

114. $(27m^3-n^3)\div(3m-n)$.

115. $(16x^2-y^2)\div(4x+y)$.

116. $(625u^4-v^4)\div(5u-v)$.

117. $(a^6b^6-x^6)\div(ab-x)$.

118. $(d^4-x^4y^4)\div(d+xy)$.

119. $(1+a^5y^5)\div(1+ay)$.

120. $[(a+b)^2-c^2]\div(a+b-c)$.

121. $[x^2-(a-b)^2]\div(x-a+b)$.

122. $[(a+b)^2-(c-d)^2]\div(a+b+c-d)$.

123. $[(x+y)^5+z^5]\div(x+y+z)$.

124. $[a^7+(y+z)^7]\div(a+y+z)$.

125. $[(m+n)^3-p^3]\div(m+n-p)$.

126. $[(a-b)^4-x^4]\div(a-b+x)$.

127. $[d^6-(e-h)^6]\div(d-e+h)$.

128. $[f^4-(x-y)^4]\div(f+x-y)$.

129. $[m^4n^4-(x+y)^4]\div[mn-(x+y)]$.

130. $(\frac{1}{4}a^2-\frac{1}{9}d^2)\div(\frac{1}{2}a+\frac{1}{3}d)$.

131. $(\frac{1}{8}x^3+y^3)\div(\frac{1}{2}x+y)$.

132. $(\frac{1}{2}\frac{1}{4}\frac{1}{8}a^5-m^5)\div(\frac{1}{8}a-m)$.

133. $(\frac{1}{1}\frac{1}{2}\frac{1}{5}d^3+\frac{1}{6}\frac{1}{4}h^3)\div(\frac{1}{5}d+\frac{1}{4}h)$.

134. $(1-\frac{8}{2}\frac{}{7}x^3)\div(1-\frac{2}{3}x)$.

135. $(x^6+y^3)\div(x^2+y)$.

136. $(a^{10}-m^{15})\div(a^2-m^3)$.

137. $(y^{12}-z^4)\div(y^3+z)$.

138. $(m^8-n^{12})\div(m^2-n^3)$.

Ex. 8.

Resolution into Factors.

Resolve into lowest factors:

1. a^2+2ab.

2. $3x^2+2x-1$.

3. $36x^2-49a^2$.

4. x^4-64.

5. $\frac{x^2}{a^2}+\frac{a^2}{x^2}-2$.

6. x^2-x-6.

7. $(x+1)^6-1$.

8. x^5+1.

9. x^2+8x+7.
10. $4x^2+32x+39$.
11. $4(ab+cd)^2-(a^2+b^2-c^2-d^2)^2$.
12. $a^2-b^2+2bc-c^2+d^2-2ad$.
13. $x^3+(a+b+c)x^2+(bc+ca+ab)x+abc$.
14. $x^5-x^4y+x^3y^2-x^2y^3+xy^4-y^5$.
15. $(a^2+ab+b^2)x^2+2(a^2+b^2)xy+(a^2-ab+b^2)y^2$.
16. $a^3-2a^2b+a^2-4a+8b-4$.
17. $(a+b)^4+4a^2(a+b)^2+16a^4$.
18. $(x+1)^2-5x-29$.
19. $1-7x-30x^2$.
20. x^6-64.
21. $3a^3+6a^2b$.
22. x^2+x-6.
23. $15x^2+19x-56$.
24. $81a^4b^4-256c^4$.
25. x^2-4y^2.
26. $x^4+\frac{288}{169}x^2+1$.
27. $6x^2-13x+6$.
28. $64x^6-729y^6$.
29. $x^2+23x+22$.
30. $9x^2-25y^2$.
31. x^2-6x+5.
32. $x^4+x^2+\frac{1}{4}$.
33. $x^2+24x+144$.
34. x^3-8.
35. $7x^6-21x^4$.
36. x^2-12x.
37. x^4-4.
38. $x^2-16x+63$.
39. $ac-ad+bc-bd$.
40. $1+4x-96x^2$.
41. $4x^2-(y-z)^2$.
42. x^3+27.
43. a^3b^3-1.
44. $(x+y)^7-(x^7+y^7)$.
45. $-4x^7+64x^6$.
46. $x^{2n}+(a+b)x^ny^n+aby^{2n}$.
47. $x^4-(4a^2+b^2)a^2x^2+4a^6b^2$.
48. $x^3-27y^3+6y(x^2+3xy+9y^2)$.
49. $a^5-a^4b+a^3b^2-a^2b^3+ab^4-b^5$.
50. $x^2-b(c-a)x-ca(b^2+bc+ba+ca)$.
51. $(x+y)^3+z^3-3xy(x+y+z)$.

52. $(a^2+ab+b^2)x^2-2abxy-(a^2-ab+b^2)y^2$.

53. $(x^2-yz)^3+8y^3z^3$.

54. $a^4+b^4-c^4-2a^2b^2$.

55. $x^4-(a+b)x^3+ab(a+b)x-a^2b^2$.

56. $(a+b)^2+(c+d)^2+2(ac+ad+bc+bd)$.

57. $x^4y^6+20x^2y^3z+96z^2$.

58. $a^2cx-abdx-abcy+b^2dy$.

59. $a^2+b^2+c^2+2ab-2ac-2bc$.

60. $x^3+(a+b)x-2a^2-ab$.

61. $36x^4+96x^2+55$.

62. $(x+y+z)^3-x^3-(y^3+z^3)$.

63. $x^4-2xy^3+y^4$.

64. $x^2+\frac{1}{x^2}-\frac{19}{9}$.

65. ax^3+x^2+ax+1.

66. $x^{15}+1024y^{10}$.

67. $(a+b)^3-a^3-b^3$.

68. x^4-11x^2+1.

69. $5x^3-10x^2-5x$.

70. $x^{2m}+2+\frac{1}{x^{2m}}$.

71. $\frac{x^2}{a^2}+\frac{2a^2}{x^2}+3$.

72. $x^4+x^2y^2+y^4$.

73. $x^2-9xy+20y^2$.

74. $6xy-x^2-9y^2$.

75. a^8-81.

76. $(a-2b)^2-4b^2$.

77. $x^9+x^6+x^3-3$.

78. $3x^2+9xy-\frac{10}{3}y^2$.

79. $x^2+ax+bx+ab$.

80. $x^4+y^4-\frac{x^2y^2}{4}$.

81. $x^2+26x+120$.

82. x^4+x^2+1.

83. $a^4b^{12}-c^8$.

84. x^6+x^3-2.

85. $x^4-18x^2y^2+y^4$.

86. $4x^2+4ax$.

87. $x^2+24x+143$.

88. $6a^2x-a^2x^2-9a^4$.

89. a^6-b^6.

90. $x^4-(2a^2+b^2+c^2+2ca+2ab)x^2+[a^2+a(b+c)+bc]^2$.

91. $ax^4+bx^3+mbx-am^2$.

92. $ax^{m+2}y^{n+3} + bx^{m+1}y^{n+2} + cx^{m}y^{n+1}$.

93. $x^2 + c(a+b)x - ab(a-c)(b+c)$.

94. $(a+b)^2 + (a-b)^2 - 2(a^2-b^2)$.

95. $(1+a)^2(1+b^2) - (1+b)^2(1+a^2)$.

96. $x^3 - (3bc + ca + ab)x + 3abc(b+c)$.

97. $11x^5 + 209x^3y + 528xy^2$.

98. $(a^2-b^2)x^2 + 4abxy + (b^2-a^2)y^2$.

99. $x^3 - y^3 + 2x(x^2-y^2) + 3y^2(x-y)$.

100. $(1-abc)(1+2abc) - (a^2+b^2+c^2)(1-abc)$.

101. $x^3 + x^2y + x^2z - xyz - y^2z - yz^2$.

102. $a^4 - 2(b^2+c^2)a^2 + (b^2-c^2)^2$.

103. $(a^2-b^2)x^2 + 2(a^2+b^2)xy + (a^2-b^2)y^2$.

104. $xy(a^2+b^2) - ab(x^2+y^2)$.

105. $abx^2 + (a^2+b^2)x + ab$.

106. $x^2 + (a+b-c)x - ac - bc$.

107. $2x^{10}y^4 - 60x^5y^2 + 450$.

108. $4abx^2 - 2(a^2+b^2)xy + aby^2$.

109. $(x+y)^2(b+c) - (x+y)(b+c)^2$.

110. $x^2 - 2(a^2+b^2)x + (a^2-b^2)^2$.

111. $x^2 - (y+2)x - 2(y^2+7y+12)$.

112. $6a^4x^2 + a^3x - 6a^3x^3 - a^2x^2$.

113. $8 - 50a^2b^2$.

114. $x^2 + 4xy - 12y^2$.

115. $(a+b)^2 - 4b^2$.

116. $x^2 + 17x + 72$.

117. $x^2 + \frac{5}{6}x + \frac{1}{6}$.

118. $9x^2 + 36$.

119. $\frac{x^2}{a^2} + \frac{a^2}{x^2} + 1$.

120. $x^6 + 19x^3 - 216$.

121. $x^2 - 22x + 121$.

122. $60x^2 - 4x - 45$.

123. $ax^2 + (a+b)x + b$.

124. $4a^2b^2 - (a^2+b^2-c^2)^2$.

125. $20x^2 - 9xy + y^2$.

126. $x^4 + \frac{288}{169}x^2 + 1$.

127. $x^2 + x - 56$.

128. $x^4 - 2x^3y + 3xy^3$.

129. $x^{16} - a^{16}$.

130. $49x^2 - 81y^4$.

131. $x^4 - 13x^2 + 40$.

132. $x^2 + \frac{9}{20}x + \frac{1}{20}$.

133. $x^2 + x - 132$.

134. $x^2 - 20x$.

135. $4x^2y^4 - 16x^4y^2$.

136. $3 - 12x^2$.

137. $x^4 - 2a^2x^2 + a^4x^2$.

138. $4x^2 - 4x - 3$.

139. $a^3 + a^2b + ab^2 + b^3$.

140. $x^{16} - 25^6$.

141. $x^3 + (a - b + c)x^2 - (bc - ca + ab)x - abc$.

142. $(x + y)(b + c - a) - (x + y)(a + b + c)$.

143. $a^2 + 2a(b + c) + b^2 + 2bc + c^2$.

144. $256x^4y^4 - 288x^2y^2 + 81$.

145. $9x^4 + 3x^2y^2 + 4y^4$.

146. $x^2 + (a + 2b)x + ab + b^2$.

147. $ax^3 + (a + b)x^2 + (a + b)x + a$.

148. $b(x^3 - a^3) + ax(x^2 - a^2) + a^3(x - a)$.

149. $a^4 + b^4 + c^4 - 2b^2c^2 - 2c^2a^2 - 2a^2b^2$.

150. $(a + b)x^2 + (a - 2b)xy - 3by^2$.

151. $(a^2 + b^2 - 20)^2 - (a^2 - b^2 + 12)^2$.

152. $(x - y)^3 - z^3 + 3xy(x - y - z)$.

153. $a^2b - bx^2 + a^2x - x^3$.

154. $b^2c^2x^2 - bca^2x - bcx + a^2$.

155. $8x^3 - 6xy(2x + 3y) + 27y^3$.

156. $x^2 + (a - b)x - 2a^2 - 2ab$.

157. $abxy + acx + bcy + b^2y^2$.

158. $5x^4 + 85x^3y + 210x^2y^2$.

159. $x^3 - (a + b + c)x + ab + ac$.

160. $80x^{22} - 11x^{11}y^2 - 60y^4$.

161. $1 - a^2 + x - ab + \frac{1}{4}(x^2 - b^2)$.

162. $a^2+(ac-b^2)x^2-bcx^3$.

163. $x^2-(2a-1)x-6-a+a^2$.

164. x^2+xy.

165. $(x^2-yz)^3+8y^3z^3$.

166. $30x^2y+3x^4+75y^2$.

167. x^2-2x-3.

168. x^4-7x^2+1.

169. $16x^4-81y^4$.

170. $x^8-16a^4x^4$.

171. x^4+64y^4.

172. x^6-x^3-110.

173. $16+44x-42x^2$.

174. $2x^2+x-1$.

175. $a^3x^2y+27x^2y^4$.

176. $7x^3+96x^2-103x$.

177. x^2-6x+8.

178. $1+15x+14x^2$.

179. $x^2+15x+54$.

180. x^4-14x^2+1.

181. $x^4+\frac{697}{861}x^2+1$.

182. x^2-x-a^2-3a-2.

183. $63x^2-21x-42$.

184. $2x^3-x^2-4x+2$.

185. $x^3+x^2y+xy^2+y^3$.

186. $3x^2-15x-42$.

187. $75x^5-35x^3-10x^4$.

188. $x^2-30x+200$.

189. $(a+b)^2-(a-b)^2$.

190. $x^2+11xy+30y^2$.

191. $9x^2+27xy-70y^2$.

192. $x^2-(a^2-b^2)^2$.

193. $(a+b)^2+2(ac+bc)+c^2$.

194. $ab(x^2-y^2)-(a^2-b^2)xy$.

195. $2x^5+2x^4+3x^3+3x^2+4x+4$.

196. $x^2+(a+2b)x+2ab$.

197. $b(x^3+a^3)+ax(x^2-a^2)+a^3(x+a)$.

198. $x^4-16a^2x^2+32a^3x-16a^4$.

199. $2x^2+4ax+6bx+12ab$.

200. $x^4-2(a^2+b^2)x^2-c^4+2(a^2-b^2)c^2+4a^2b^2$.

201. $3(1-x^2)+\frac{a^2}{3}-2a$.

202. $abx^2-(a^2-b^2)x-ab$.

203. $(x+y)(b+c)+(x+y)(b-c)$.

204. $ab(x^2+y^2)+(a^2+b^2)xy$.

205. $a^4-2(b^2-c^2)a^2+b^4-2b^2c^2+c^4$.

206. $x^3-(2a-b)x^2-(2ab-a^2)x+a^2b$.

207. $x^3+4y^3-9z^3-4xy$.

208. $8x^3+12ax+10bx+15ab$.

209. $81(a-b)^2-63(a-b)x-60x^2$.

210. $(x^2+x+1)^2-2x^2-4x-3$.

211. $4x^4+30x^2y^2-100y^4$.

212. $x^3+2ax+3bx+6ab$.

213. $16a^2b^2-(a^2+4b^2-c^2)^2$.

214. $3x^4y^3z^3-6x^3y^4z^3+12x^3y^2z^4$.

215. $x^2-(2a+b)x+2ab$.

216. $28x^2-84x+35$.

217. $x^4-11x^2y^2+y^4$.

218. x^4-171x^2+1.

219. $(a^2+b^2)^2-4a^2b^2$.

220. $(a-b)^3-a^3+b^3$.

221. $x^2+13x-300$.

222. $5-70x^2y^2+245x^4y^4$.

223. $5x^2-15xy-\frac{5}{9}y^2$.

224. x^4-167x^2+1.

225. $(x^2+3)^2-16x^2$.

226. $9x^2y^2-3xy^3-6y^4$.

227. $x^2+2ax+a^2-b^2$.

228. $(3x-2)^2-(2x-3)^2$.

229. $x^2+9xy-10y^2$.

230. $28x^2y^2+3xy-1$.

231. x^3+2x^2+4x+8.

232. $(ax+by)^2-(ax-by)^2$.

233. $49x^2+49xy+6y^2$.

234. $18a^2b^2+19ab+1$.

235. $ax^2+(a-b)x-b$.

236. $x^2-8xy+12y^2$.

237. $x^8-82x^4y^4+81y^8$.

238. x^6+11x^3-620.

239. $a^3+ab^2-2a^2b-2b^3$.

240. $28x^2+14x-84$.

241. $7x^{10}-21x^5+14$.

242. $21x^2+26xy-15y^2$.

243. $x^2-b(a-c)x-ac(a+b)(b+c)$.

244. $a^2bc+b^2c^2-ab^3-ac^3$.

245. $13x^4-118x^2y^2+240y^4$.

246. $a(x^2+y^2)+(a^2+1)xy$.

247. $x^4+(2-m^2)x^2y^2+y^4$.

248. $x^2(x-a)-bx(x-a)$.

249. $(x^2+5x)^2-8x^2-40x-84$.

250. $x^4+2x^2y^2+(1-m^2)y^4$.

251. $99x^4-13x^2y-132y^2$.

252. $x^4y^2 - 17x^2yz^2 + 16z^4$.

253. $x^2 - c(7x + ab) - a^2b^2 + 12c^2$.

254. $x^4 - (2 + m^2)x^2y^2 + y^4$.

255. $ab(x^2 + y^2) - (a^2 + b^2)xy$.

256. $a^3 - 4a^2b + 4ab^2 - 9ax^2$.

257. $a^2x^4 - (b^2 - 2ac)x^2 + c^2$.

258. $x^4 - (a^2 + b^2)x^2 + a^2b^2$.

259. $xy(1 + z^2) + z(x^2 + y^2)$.

260. $(x + y)^2 - 7(x + y)z - 30z^2$.

261. $abx^2 + c(a - b)x - c^2$.

262. $(x^2 + 6x + 7)^2 - (x + 3)^2$.

263. $9(x + 1)^2 - 3(x + 1) - 6$.

264. $x^4 - 2abx^2 - a^4 - a^2b^2 - b^4$.

265. $45x^4 + 98x^2y^6 - 75y^{12}$.

266. $2a^3 - a^2b + 4abx - 2b^2x$.

267. $x^2 + 2bx - a^2 + b^2$.

268. $x^2 - 2bx + b^2 - a^2$.

269. $99x^2 - 297x + 220$.

270. $a^3b - 2a^2b^2 - 2ab^3$.

271. $x^8y^4 + 15x^4y^2 + 36$.

272. $192x^4 - 480x^2 - 132$.

273. $x^4 - 63x^2y^2 - 64y^4$.

274. $11x^2 - 54xy^2 + 63y^4$.

275. $x^4 + 13x^2y + 36y^2$.

276. $3x^2 + 51xy + 90y^2$.

277. $a^3 - a^2b - 2ab^2 + 2b^3$.

278. $\frac{3}{5}(a + b)^2 - \frac{5}{8}(a - b)^2$.

279. $24a^2x^2 - 14ax^3 - 3x^4$.

280. $3x^4 + 2x^3 - 21x - 14$.

281. $a^7 - a^5b^2 + 2a^4b^3 - a^3b^4$.

282. $abx^2 + (b - a)xy - y^2$.

283. $x^3 - a^2x^2 - b^2x + a^2b^2$.

284. $a^{12} - a^8b^4 - a^4b^8 + b^{12}$.

285. $x^3 + bx^2 - a^2x - a^2b$.

286. $(x - 2y)x^3 - (y - 2x)y^3$.

287. $2a + (a^2 - 4)x - 2ax^2$.

288. $5x^4 + 4x^3 - 20x - 125$.

289. $a^2 + (ac - b^2)x^2 + bcx^3$.

290. $36x^4 - 12x^3 - 35x^2$.

291. $(a + b)^3 + (a - b)^3$.

Ex. 9.

Highest Common Factor.

Find the highest common factor of:

1. $4(x+1)^2$ and $6(x^2-1)$.
2. $18(a^2-b^2)$ and $27(a+b)^2$.
3. $6(x+1)^3$ and $9(x^2-1)$.
4. $12(x^3+y^3)^2$ and $8(x^4-y^4)$.
5. (a^3-x^3) and a^2-x^2.
6. $(a^2x-ax^2)^2$ and $ax(a^2-x^2)^2$.
7. a^6-b^6 and a^4-b^4.
8. x^5-y^5 and x^3-y^3.
9. $12a^4c^2d-48b^4d$ and $15f(a^2c+2b^2)^2$.
10. $6ac+9bc-2ad-3bd$ and $10af+16ag+15bf+24bg$.
11. $bc-2b^2-ac+2ab$ and $2b^2-2ab+3bc-3ac$.
12. $2a^2+5a-4ab-10b$ and $4a^2-6ac+10a-15c$.
13. $16a^2-9b^2$ and $16ab+12b^2+12ac+9bc$.
14. $25x^2-1$ and $35x^2-7x+5ax-a$.
15. $21a^6x^5y^3-28a^4x^7y^5-42a^3x^8y^4$ and $33a^3x^5-44ax^7y^2-66x^8y$.
16. $x^2+2xy+y^2$ and x^3-y^3.
17. $a^3+3a^2b+3ab^2+b^3$ and $a^2+2ab+b^2$.
18. $4x^2-12xy+9y^2$ and $4x^2-9y^2$.
19. $a^2+2ab+b^2$ and a^2-b^2.
20. $x^2+8x+15$ and $x^2+9x+20$.
21. $x^2-9x+14$ and $x^2-11x+28$.
22. $x^2+2x-120$ and $x^2-2x-80$.

23. $x^2 - 15x + 36$ and $x^2 - 9x - 36$.

24. $x^2 - 1$ and $x^2 + 4x + 3$.

25. $6x^2 + 5x - 6$ and $6x^2 - 13x + 6$.

26. $12x^2 - 17ax + 6a^2$ and $9x^2 + 6ax - 8a^2$.

27. $6x^2 + 13x + 6$ and $8x^2 + 6x - 9$.

28. $x^3 + 4x^2 - 5x$ and $x^3 - 6x + 5$.

29. $x^3 + 2x^2 - 3x$ and $2x^3 + 5x^2 - 3x$.

30. $14 - 34x + 12x^2$ and $42a - 4ax - 6ax^2$.

31. $x^3 + 2x^2 + 2x + 1$ and $x^3 - 2x - 1$.

32. $x^3 - 3x^2 + 7x - 21$ and $2x^4 + 19x^2 + 35$.

33. $6x^3 - 6xy^2 + 2x^2y - 2y^3$ and $12x^2 - 15xy + 3y^2$.

34. $a^4 + 7a^3 + 7a^2 - 15a$ and $a^3 - 2a^2 - 13a + 110$.

35. $3x^3 - 3x^2y + xy^2 - y^3$ and $4x^3 - x^2y - 3xy^2$.

36. $3a^3 - 3a^2b + ab^2 - b^3$ and $4a^2 - 5ab + b^2$.

37. $x^4 + a^3x - ax^3 - a^4$ and $x^3 - a^3$.

38. $x^4 - 2x^3y + 2xy^3 - y^4$ and $x^4 - 2x^3y + 2x^2y^2 - 2xy^3 + y^4$.

39. $20x^4 + x^2 - 1$ and $75x^4 + 15x^3 - 3x - 3$.

40. $x^4 - 6x^3 + 13x^2 - 12x + 4$ and $x^5 - 4x^4 + 8x^3 - 16x^2 + 16x$.

41. $2x^4 - 11x^2y^2 + 12y^4$ and $3x^6 - 48x^2y^4$.

42. $2a^5 - 4a^4 + 8a^3 - 12a^2 + 6a$ and $3a^5 - 3a^4 - 6a^3 + 9a^2 - 3a$.

43. $36x^6 - 18x^5 - 27x^4 + 9x^3$ and $27x^2y^3 - 18x^4y^3 - 9x^3y^3$.

44. $3x^4 - x^2y^2 - 2y^4$ and $10x^4 + 15x^3y - 10x^2y^2 - 15xy^3$.

45. $a^6 + 4a^5 - 3a^4 - 16a^3 + 11a^2 + 12a - 9$.
and $6a^5 + 20a^4 - 12a^3 - 48a^2 + 22a + 12$.

46. $2x^3 - 16x + 6$ and $5x^6 + 15x^5 + 5x + 15$.

47. $a^6 - b^4a^2 - ba^4 + b^5$ and $a^3 + b^2a^6 + ba^2 + b^3$.

48. $4a^5b - 4a^3b^3 + 4a^2b^4 - ab^5$ and $24a^5 + 16a^4b - 36a^3b^2 - 12a^2b^3 + 8ab^4$.

49. $35x^3+47x^2+13x+1$ and $42x^4+41x^3-9x^2-9x-1$.

50. $x^6-3x^5+6x^4-7x^3+6x^2-3x+1$
and $x^6-x^5+2x^4-x^3+2x^2-x+1$.

51. $2x^4-6x^3+3x^2-3x+1$ and $x^7-3x^6+x^5-4x^3+12x-4$.

52. x^8-1 and $x^{10}+x^9+x^8+2x^7+2x^4+2x^3+x^2+x+1$.

53. $2a^4+3a^3x-9a^2x^2$ and $6a^4x-17a^3x^2+14a^2x^3-3ax^4$.

54. $x^5+3x^4-8x^2-9x-3$ and $x^5-2x^4-6x^3+4x^2+13x+6$.

55. $6x^5-4x^4-11x^3-3x^2-3x-1$
and $4x^4+2x^3-18x^2+3x-5$.

56. $x^4-ax^3-a^2x^2-a^3x-2a^4$ and $3x^3-7ax^2+3a^2x-2a^3$.

57. $x^2+(a-b)x-ab$ and $x^2+(a+b)x+ab$.

58. $x^4-(a^2+b^2)x^2+a^2b^2$
and $x^4-(a+b)^2x^2+2ab(a+b)x-a^2b^2$.

59. $2x^3+(2a-9)x^2-(9a+6)x+27$ and $2x^2-13x+18$.

60. $x^4-px^3+(q-1)x^2+px-q$
and $x^4-qx^3+(p-1)x^2+qx-p$.

61. $x^4-2a(a-b)x^3+(a^2+b^2)(a-b)x-a^2b^2$
and $x^4-(a-b)x^3+(a-b)b^2x-b^4$.

62. $ax^6+(a+b)x^5+(a+b+c)x^4+(a+b+c+d)x^3$
$+(b+c+d)x^2+(c+d)x+d$ and $ax^5+(a+b)x^4$
$+(a+b+c)x^3+(a+b+c)x^2+(b+c)x+c$.

63. $3x^2-(3c+d-3)x-3c-d$
and $2x^2+(2a+b+2)x+2a+b$.

64. x^4-a^4, x^3+a^3, and x^2-a^2.

65. $a^3+a^2b-ab^2-b^3$, $a^3-3ab^2+2b^3$,
and $a^3-2a^2b-ab^2+2b^3$.

66. x^2-2a^2-ax, x^2-6a^2+ax, and x^2-8a^2+2ax.

67. a^5+b^5, a^4-b^4, and a^3+b^3.

68. $3x^3-7x^2y+5xy^2-y^3$, $x^2y+3xy^2-3x^3-y^3$,
and $3x^3+5x^2y+xy^2-y^3$.

Ex. 10.

Lowest Common Multiple.

Find the lowest common multiple of:

1. $25a^3b^4c^5$ and $20a^5b^2c^6$.
2. $a(a+b)$ and $b(a+b)$.
3. $12a(b-1)$ and $12b(b-1)$.
4. $15b(a+b)$ and $18b(a-b)$.
5. $(a+b)(c+d)$ and $(a+b)(c-d)$.
6. a^2-b^2 and $(a-b)^2$.
7. $3(a+x)$ and $4(a^2-x^2)$.
8. a^2bx-ab^2y and $abx+b^2y$.
9. b^2 and $b(b+ax)$.
10. x^2-3x-4 and x^2-x-12.
11. x^2-1 and x^2+4x+3.
12. $6x^2+5x-6$ and $6x^2-13x+6$.
13. $12x^2+5x-3$ and $6x^3+x^2-x$.
14. x^3+5x^2+7x+2 and x^3+6x+8.
15. $12x^2-17ax+6a^2$ and $9x^2+6ax-8a^2$.
16. $a^3-9a^2+23a-15$ and a^2-8a+7.
17. $a^3+2a^2b-ab^2-2b^3$ and $a^3-2a^2b-ab^2+2b^3$.
18. $15x^5+10x^4y+4x^3y^2+6x^2y^3-3xy^4$ and $12x^3y^2+38x^2y^3+16xy^4-10y^5$.
19. $a^5+a^4b+a^3b^2+a^2b^3+ab^4+b^5$ and $a^5-a^4b+a^3b^2-a^2b^3+ab^4-b^5$.
20. mn, mp, and pq.
21. $4x^3y^2z$, $6xy^3z^2$, and $18x^2yz^3$.
22. $x+y$, $x-y$, and x^2-y^2.
23. x^2-y^2, $(x-y)^2$, and $(x+y)$.
24. $6a$, $2(a+1)$, and $2(a+2)$.
25. a, $(a+2b)$, and a^2-4b^2.
26. a, $2a-1$, and $4a^2-1$.
27. $(a+b)$, $2a$, and $2a(a-b)$.
28. $x-1$, x^2+x+1, and x^3-1.

29. x^2+3x+2, x^2+4x+3, and x^2+5x+6.

30. $x^2+5x+10$, $x^3-19x-30$, and $x^3-15x-50$.

31. x^2+2x-8, x^3+3x^2-x-3, and x^3+4x^2+x-6.

32. $6x^2-13x+6$, $6x^2+5x-6$, and $9x^2-4$.

33. $a^3-6a^2+11a-6$, $a^3-9a^2+26a-24$, and $a^3-8a^2+19a-12$.

34. x^2-1, x^2+1, x^4+1, and x^8-1.

35. x^2-1, x^3+1, x^3-1, and x^6+1.

36. $x^2-(a+b)x+ab$, $x^2-(b+c)x+bc$, and $x^2-(c+a)x+ac$.

37. $2x^3+(2a-3b)x^2-(2b^2+3ab)x+3b^3$, and $2x^2-(3b-2c)x-3bc$.

Ex. 11.

Reduction of Fractions to Lowest Terms.

Simplify the following fractions:

1. $\dfrac{(15m^2n^2p)(7p^2n)}{(14p^3)(5m^2q)}$.

2. $\dfrac{10a^2-2ab}{15ab-3b^2}$.

3. $\dfrac{39a^2b^3-36ab^3}{65a^3bc-60a^2bc}$.

4. $\dfrac{42a^3-30a^2x}{35ax^2-25x^3}$.

5. $\dfrac{16a^3b-20a^2c}{28a^2b^3c^2-35ab^3c^3}$.

6. $\dfrac{156a^3b^2-104a^2b^3}{351a^2bx-234ab^2x}$.

7. $\dfrac{21a^2b^2-35b^3c-12a^3c+20abc^2}{18a^2c^2-21a^3b-30bc^3+35ab^2c}$.

8. $\dfrac{40a^2b^4-32a^3bx^2-5b^3x^3+4ax^5}{4ab^2x^3-36a^4x^2-5b^5x+45a^3b^3}$.

9. $\dfrac{14a^3x^5+9a^2b^6-6b^4x^4-21a^5b^2x}{12a^4b^4-28a^7x+21a^3bx^4-9b^5x^3}$.

10. $\dfrac{a^3+(1+a)\,ab+b^2}{a^3+(1-3c^2)\,ab-3a^3c^2}$.

11. $\dfrac{ab(ab-3c^2)-c(3a^3-b^3)}{ac(ac-4b^2)-b(4a^3-c^3)}$.

12. $\dfrac{6x^2-3y-2x^4+x^2y}{9x^2-3y^2-3x^4+x^2y^2}$.

13. $\dfrac{2x^2z^2-3x^2y^2-2y^2z^2+3y^4}{3x^2z^2+2y^4-2x^2y^2-3y^2z^2}$.

14. $\dfrac{6a^2c^2-2a^4+18bc^2-6a^2b}{4a^4+2a^2c^2+12a^2b+6bc^2}$.

15. $\dfrac{15a^3-28bc+12a^2b-35ac-18a^2c+42c^2}{45ab^2+36b^3+20a^3-54b^2c+16a^2b-24a^2c}$.

16. $\dfrac{35a^5+24a^2b^2-21a^3b+15a^3b^2-40a^4b-9ab^3}{28a^3b^2-42a^4-18a^2b^2-32a^2b^3+48a^3b+12ab^4}$.

17. $\dfrac{2x-1}{4x^2-1}$.

18. $\dfrac{3x+1}{9x^2-1}$.

19. $\dfrac{8x^3+1}{64x^6-1}$.

20. $\dfrac{x^3+3x^2}{x^2-9}$.

21. $\dfrac{x^2y-xy}{x^2-1}$.

22. $\dfrac{x^4-y^3}{x^3+xy}$.

23. $\dfrac{9x^5-4x}{6x^2y^2-4y^2}$.

24. $\dfrac{x^6y^4z^2-x^4y^8}{x^6y^2z^4-x^2y^{10}}$.

25. $\dfrac{3ax+9x^2}{4a^2-36x^2}$.

26. $\dfrac{x^3+x^2y}{x^2+2xy+y^2}$.

27. $\dfrac{x^3+2x^2}{x^2+4x+4}$.

28. $\dfrac{12x^2+8xy}{9x^2+12xy+4y^2}$.

29. $\dfrac{3a^2b^2-6b^2c}{a^4-4a^2c+4c^2}$.

30. $\dfrac{7x^2+3}{245x^5+210x^3+45x}$.

31. $\dfrac{15a^2-27ab}{25a^2-90ab+81b^2}$.

32. $\dfrac{a^4-b^4}{a^2+2ab+b^2}$.

33. $\dfrac{x-xy+z-zy}{1-3y+3y^2-y^3}$.

34. $\dfrac{a^2x+abx}{a^3+3a^2b+3ab^2+b^3}$.

35. $\dfrac{20a^5x^2+16a^3bx^2}{75a^4b+120a^2b^2+48b^3}$.

36. $\dfrac{x^4+(2b^2-a^2)x^2+b^4}{x^4+2ax^3+a^2x^2-b^4}$.

37. $\dfrac{x^2+(a+b+c)x+(a+b)c}{a^2+2ab+b^2-x^2}$.

38. $\dfrac{x^2+(b-c-a)x-(b-c)a}{x^2-b^2+2bc-c^2}$.

39. $\dfrac{ax^2+(bc-ab-ac)x+abc-b^2c}{b^2c^2-2abc^2+a^2c^2-a^2x^2}$.

40. $\dfrac{a^2bc-b^3c+2b^2c^2-bc^3}{4a^2b^2-(a^2+b^2-c^2)^2}$.

41. $\dfrac{x^2+y^2+z^2+2xy+2xz+2yz}{x^2-y^2-z^2-2yz}$.

42. $\dfrac{x^2+3x+2}{x^2+6x+5}$.

43. $\dfrac{x^2+10x+21}{x^2-2x-15}$.

44. $\dfrac{x^2-4x+4}{x^2-5x+6}$.

45. $\dfrac{x^2-3x+2}{x^2-1}$.

46. $\dfrac{x^2-7x+12}{x^2-8x+15}$.

47. $\dfrac{x^2-x-20}{x^2+x-30}$.

48. $\dfrac{a^2-9ab+14b^2}{a^2-ab-2b^2}$.

49. $\dfrac{2a^2-ab-3b^2}{2a^2-5ab+3b^2}$.

50. $\dfrac{x^2-9}{x^2-6x+9}$.

51. $\dfrac{2x^3+5x^2-12x}{7x^3+25x^2-12x}$.

52. $\dfrac{4a^2-7ab+3b^2}{5a^2-3ab-2b^2}$.

53. $\dfrac{2x^3+5x^2-3x}{x^3+2x^2-3x}$.

54. $\dfrac{3x^2-10xy+8y^2}{5x^2-13xy+6y^2}$.

55. $\dfrac{12y^3-7y^2-12y}{9y^3+6y^2-24y}$.

56. $\dfrac{6a^2x+ax-x}{3a^2+2a-1}$.

57. $\dfrac{x^4-x^3-x+1}{x^4-2x^3-x^2-2x+1}$.

58. $\dfrac{a^5-a^4b-ab^4+b^5}{a^4-a^3b-a^2b^2+ab^3}$.

59. $\dfrac{bx+2}{2b+(b^2-4)x-2bx^2}$.

60. $\dfrac{(x+y)^7-x^7-y^7}{(x+y)^5-x^5-y^5}$.

Ex. 12.

Addition and Subtraction of Fractions.

1. $\dfrac{a}{m}+2m$.

2. $\dfrac{2x}{x-a}+1$.

3. $2-\dfrac{6a}{3a+2b}$.

4. $\dfrac{a}{2b}-\dfrac{b}{3a}$.

5. $\dfrac{5a}{x^2}-\dfrac{2b}{xy}$.

6. $\dfrac{x}{10y}-\dfrac{3y}{15x}$.

7. $\dfrac{n}{a^2}+\dfrac{x}{5a}$.

8. $\dfrac{a+b}{a}+\dfrac{a-b}{a}$.

9. $\dfrac{a+b}{a}-\dfrac{a+b}{b}$.

10. $\dfrac{a}{b}+\dfrac{a+b}{2b}$.

11. $\dfrac{a+b}{3a}-\dfrac{b}{12a}$.

12. $\dfrac{5a-2b}{2}+\dfrac{3a-5b}{3}$.

13. $\dfrac{15a+4b}{12}-\dfrac{3b-22a}{9}$.

14. $\dfrac{3a+2b}{a}+\dfrac{2a^2-2b^2}{ab}$.

15. $\frac{7}{12}x+\frac{10}{21}y-\frac{1}{7}y-\frac{1}{4}x$.

16. $\frac{4}{5}a-\frac{3}{10}a-\frac{4}{7}b+\frac{1}{14}b$.

17. $\dfrac{2x}{15}+\dfrac{5y}{12}-\dfrac{x}{5}-\dfrac{3y}{4}$.

18. $\dfrac{4a-23b}{4}-\dfrac{4a-25b}{6}+\dfrac{19b-3a}{12}$.

19. $\dfrac{3x-2y}{3}-\dfrac{4y+2x}{5}+\dfrac{22y-9x}{15}$.

20. $\dfrac{29a}{90}-\dfrac{3a-20b}{60}+\dfrac{3a-10b+5c}{20}-\dfrac{4a-2b+3c}{12}$.

21. $\frac{4a-3b+2c}{9}-\frac{b+c-3a}{4}+\frac{b-2a}{2}-\frac{2a-b}{3}+\frac{c-b+2a}{6}$.

22. $\frac{a-4b-3c}{4}-\frac{c-a}{2}-\frac{3a-4b-6c}{3}+\frac{2a-b-3c}{7}$
$-\frac{6b+6c-2a}{21}$.

23. $\frac{3a+2b}{a}+\frac{2a^2-2b^2}{ab}-\frac{2a+3b}{b}$.

24. $\frac{a-b}{a}+\frac{a^2(b+1)+b^2(a+1)}{ab}-\frac{a+b}{b}$.

25. $\frac{7b-5a}{3b}-\frac{c-7a}{a}+\frac{a+5c}{5a}+\frac{11a}{6b}$.

26. $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$.

27. $\frac{a}{m}+\frac{a}{n}+\frac{a}{p}$.

28. $\frac{a}{xy}-\frac{b}{xz}-\frac{c}{yz}$.

29. $\frac{2}{x^2}+\frac{3}{ax}-\frac{4}{bx}$.

30. $\frac{1}{2b}-\frac{1}{3b}-\frac{1}{4b}+\frac{1}{5b}$.

31. $\frac{4a}{3x}-\frac{5a}{6x}-\frac{4a}{5x}+\frac{7a}{15x}$.

32. $\frac{3a}{bc}-\frac{5a}{bd}+\frac{4a}{bf}$.

33. $\frac{2a}{5b}+\frac{8a}{3bg}-\frac{2c}{9bd}$.

34. $\frac{a-3b}{6a}+\frac{4a-b}{2b}+\frac{5a+3x}{9x}-\frac{a^2-bx}{2ax}-\frac{2a}{b}$.

35. $\frac{3a-5b}{15ab}-\frac{a-7c}{12ac}+\frac{5b-4c}{20bc}+\frac{3}{4a}+\frac{1}{b}+\frac{5}{6c}$.

36. $\frac{6a+c}{6bc}-\frac{5a-4b}{4ac}-\frac{3b-5c}{5ab}+\frac{3}{5a}-\frac{1}{6b}+\frac{5}{4c}$.

37. $\frac{2}{3a}-\frac{1}{2b}-\frac{2a+3}{6a^2}+\frac{1}{2x^2}+\frac{3a-2b}{6ab}$.

38. $\frac{1}{a-b}+\frac{1}{a+b}$.

39. $\frac{a}{a-b}+\frac{b}{a+b}$.

40. $\frac{c}{a-b}-\frac{c}{a+b}$.

41. $\frac{1}{x+y}+\frac{2y}{x^2-y^2}$.

42. $\frac{1+3a}{1-3a}-\frac{1-3a}{1+3a}$.

43. $\frac{a}{x(a-x)}-\frac{x}{a(a-x)}$.

44. $\frac{a}{2a-2b}-\frac{b}{2b-2a}$.

45. $\frac{2a+3x}{2a-3x}+\frac{2a-3x}{3x-2a}$.

46. $\frac{a-b}{2(a+b)}+\frac{a^2+b^2}{a^2-b^2}$.

47. $\frac{5x-3}{x+1}-\frac{2x^2-13x+1}{x^2-1}$.

48. $\frac{x(a+x)}{a-x}-\frac{5ax-x^2}{x-a}$.

49. $\frac{x}{a^n+1}+\frac{x}{a^n-1}$.

50. $\frac{a}{a-x}+\frac{3a}{a+x}-\frac{2ax}{a^2-x^2}$.

51. $\frac{2b-a}{x-b}+\frac{b-2a}{x+b}+\frac{3x(a-b)}{x^2-b^2}$.

52. $\frac{a-b}{b}+\frac{2a}{a-b}-\frac{a^3+a^2b}{a^2b-b^3}$.

53. $\frac{3}{x}-\frac{5}{2x-1}-\frac{2x-7}{4x^2-1}$.

54. $\frac{5a}{6b}-\frac{10a^2-17ab}{12ab-6b^2}-\frac{b}{b-2a}$.

55. $\frac{2}{x}+\frac{3}{1-2x}-\frac{2x-3}{4x^2-1}$.

56. $\frac{1}{x-2}-\frac{3}{x+2}+\frac{2x}{(x+2)^2}$.

57. $\frac{a+x}{a-x}+\frac{a-x}{a+x}-\frac{a^2-x^2}{a^2+x^2}$.

58. $\frac{1}{x+1}-\frac{2}{x+2}+\frac{1}{x+3}$.

59. $\frac{x}{x-1}-\frac{2x}{x+1}+\frac{x}{x-2}$.

60. $\frac{5}{2(x+1)}-\frac{1}{10(x-1)}-\frac{24}{5(2x+3)}$.

61. $\frac{x-1}{x+1}+\frac{x+1}{x-1}+\frac{3x-2}{x+3}+\frac{3x-4}{x-2}$.

62. $\frac{a+x}{a-x}+\frac{a-x}{a+x}+\frac{a^2+x^2}{a^2-x^2}+\frac{4ax}{a^2+x^2}$.

63. $\frac{1}{1+x}-\left[\frac{6}{1-x}-\left(\frac{2}{1+2x}-\frac{16}{2x-1}\right)\right]$.

64. $\frac{1}{x^2(x^2+y^2)}-\frac{1}{2x^2(y-x)}+\frac{1}{2x^2(x+y)}$.

65. $\frac{3}{2a-3}-\frac{2}{3+2a}-\frac{2a+15}{4a^2+9}$.

66. $\frac{a+c}{(x-a)(b-a)}-\frac{b+c}{(x-b)(b-a)}$.

67. $\frac{1}{(n+1)(n+2)}-\frac{1}{(n+1)(n+2)(n+3)}-\frac{1}{(n+1)(n+3)}$.

68. $\frac{b+c}{(a-b)(a-c)}+\frac{a+c}{(b-c)(b-a)}+\frac{a+b}{(c-a)(c-b)}$.

69. $\frac{1}{x^2-a^2}+\frac{1}{(x+a)^2}-\frac{1}{(x-a)^2}$.

70. $\frac{1}{(x-2)(x-3)}+\frac{2}{(x-1)(3-x)}+\frac{1}{(x-1)(x-2)}$.

71. $\frac{a^2-bc}{(a+b)(a+c)}+\frac{b^2-ac}{(b+c)(b+a)}+\frac{c^2-ab}{(c+a)(c+b)}$.

72. $\frac{bc}{(c-a)(a-b)}+\frac{ac}{(a-b)(b-c)}+\frac{ab}{(b-c)(c-a)}$.

73. $\frac{1}{a(a-b)(a-c)}+\frac{1}{b(b-c)(b-a)}+\frac{1}{c(c-a)(c-b)}$.

74. $\frac{x-y}{x+y}+\frac{y-z}{y+z}+\frac{z-x}{z+x}+\frac{(x-y)(y-z)(z-x)}{(x+y)(y+z)(z+x)}$.

75. $\frac{2}{x+4}-\frac{x-3}{x^2-4x+16}+\frac{x^2}{x^3+64}$.

76. $\frac{x^2+ax+a^2}{x^3-a^3}-\frac{x^2-ax+a^2}{x^3+a^3}$.

77. $\frac{2}{(x-1)^3}+\frac{1}{(x-1)^2}+\frac{2}{x-1}-\frac{1}{x}$.

78. $\frac{x^2+y^2}{xy}-\frac{x^2}{xy+y^2}-\frac{y^2}{x^2+xy}$.

79. $\frac{x^2-2x+3}{x^3+1}+\frac{x-2}{x^2-x+1}-\frac{1}{x+1}$.

80. $\frac{x-y}{x^2-xy+y^2}+\frac{1}{x+y}+\frac{xy}{x^3+y^3}$.

81. $\frac{1}{x-y}+\frac{x-y}{x^2+xy+y^2}+\frac{xy-2x^2}{x^3-y^3}$.

82. $\frac{1-2x}{3(x^2-x+1)}-\frac{2x-3}{2(x^2+1)}+\frac{1}{6(x+1)}$.

83. $\frac{x+1}{x^2+x+1}+\frac{x-1}{x^2-x+1}+\frac{2}{x^4+x^2+1}$.

84. $\frac{1}{x-3}+\frac{x-1}{x^2+3x+9}+\frac{x^2+x-3}{x^3-27}$.

85. $\frac{2a}{a^4-a^2+1}-\frac{1}{a^2-a+1}+\frac{1}{a^2+a+1}$.

86. $\frac{1}{a^2-7a+12}+\frac{2}{a^2-4a+3}-\frac{3}{a^2-5a+4}$.

87. $\frac{2+3x-5x^2}{x^3-3x^2+3x-1}-\frac{2x+5}{x^2-2x+1}+\frac{1}{x-1}$.

88. $\frac{7+3x^2}{4-x^2}-\frac{5-2x^2}{4+4x+x^2}-\frac{3-2x+x^2}{4-4x+x^2}$.

89. $\frac{x+3}{2x-1}-\frac{x^2-5}{4x^2-4x+1}-\frac{2x^3-x-3}{8x^3-12x^2+6x-1}$.

90. $\frac{3a^2-2ax+x^2}{a^3-6a^2x+12ax^2-8x^3}+\frac{a+2x}{2a^2-8ax+8x^2}$.

91. $\frac{x^2+8x+15}{x^2+7x+10}-\frac{x-1}{x+2}$.

92. $\frac{x^2-5ax+6a^2}{x^2-8ax+15a^2}-\frac{x-7a}{x-5a}$.

93. $\dfrac{(x-3a)(x-7a)}{x^2-10ax+24a^2}-\dfrac{x-5a}{x-4a}$.

94. $\dfrac{2+x}{3+x^2}-\dfrac{3-x}{9+6x+x^2}+\dfrac{1-2x^2}{(3+x)(3+x^2)}$.

95. $\dfrac{2-3x}{6-4x}+\dfrac{3-4x}{6+4x}-\dfrac{1-2x+x^2}{9-12x+4x^2}+\dfrac{2-5x^2}{36-16x^2}$.

96. $\dfrac{z^4-2z^2-3}{15z^6-17z^2-18+25z^4}-\dfrac{z^2-4z+1}{12z^4-z^2-6}$.

97. $\dfrac{5x-2}{x^2-3x-4}+\dfrac{2x+1}{x^2-x-12}-\dfrac{3x-1}{x^2+x-20}$.

98. $\dfrac{x-1}{x^2-7x+10}-\dfrac{x+2}{x^2-9x+14}-\dfrac{x-3}{x^2-12x+35}$.

99. $\dfrac{a-5x}{4a^2-20ax+25x^2}-\dfrac{3a^2-4ax+x^2}{24a^3-180a^2x+450ax^2-375x^3}$.

100. $\dfrac{a^2-bc}{(a-b)(a-c)}+\dfrac{b^2+ac}{(b-a)(b+c)}+\dfrac{c^2+ab}{(c-a)(c+b)}$.

Ex. 13.

Multiplication and Division of Fractions.

1. $\dfrac{6p^2q^3r^2}{7mx^5y^8}\times\dfrac{3q^7r^6}{5mx^6y^7}$.

2. $\dfrac{12x^2}{5y^2}\times\dfrac{10xy}{9z^2}$.

3. $\dfrac{3a^3c^3d}{4pqm}\times\dfrac{16pm}{27a^4c^3}$.

4. $\dfrac{a^2}{bc}\times\dfrac{b^2}{ac}\times\dfrac{c^2}{ab}$.

5. $\dfrac{a^2b}{x^2y}\times\dfrac{b^2c}{y^3z}\times\dfrac{x^2y^2z}{abc}$.

6. $\dfrac{5a^2b}{3cd}\times\dfrac{4b^2c}{10a^2}\times\dfrac{9c^2d}{16b^3}$.

7. $\dfrac{2a^8x^7}{3b^8}\times\dfrac{5a^4b^5}{4c^4x^6}\times\dfrac{18ab^2c^3}{25a^4x}$.

8. $\frac{5a^3b^2}{7m^2n^4} \times \frac{14a^9m^7}{25n^5b^{11}} \times \frac{5n^{11}m^6}{6a^{15}b^{13}} \times \frac{6am}{b^3n}$.

9. $\left(\frac{a}{b}\right)^2$.

10. $\left(\frac{2a}{5b}\right)^3$.

11. $\left(\frac{5xy}{7ab}\right)^4$.

12. $\left(\frac{2ab}{3cx}\right)^5$.

13. $\left(\frac{2a^2b^3x^4}{3y^5}\right)^3$.

14. $\left(\frac{a^2y^nx^{n-5}}{2b^{n-1}z^4}\right)^5$.

15. $\frac{3(2a^2x^5)^2}{4(3a^3x^4)^3}$.

16. $\left(\frac{3a^2x^3}{4bc^3}\right)^2 \times \left(\frac{2a^2c^4}{b^3x}\right)^3$.

17. $\frac{(3a^4b)^2}{(5cd)^3} \times \frac{(5c)^4}{(6a)^5} \times \frac{(4b)^6}{40a^3b^5c}$.

18. $\frac{3(a^2-b^2)}{5bc} \times \frac{10c}{9(a+b)}$.

19. $\frac{x^2-y^2}{x^2+y} \times \frac{4x}{x+y}$.

20. $\frac{(-5b^3c^5)^2}{[(-2a^2)^5]^2}$.

21. $\frac{a+b}{5} \times \frac{2}{a+b}$.

22. $\frac{x+1}{y} \times \frac{4y^2}{x^2-1}$.

23. $\frac{x^2-y^2}{x^2+y^2} \times \left(-\frac{3x^2}{4x+4y}\right)$.

24. $\frac{ab+ac}{bd+cd} \times \frac{ab-ac}{bd-cd}$.

25. $\frac{4x+4y}{3x^2-3y^2} \times \frac{5x^2-5y^2}{7x+7y}$.

26. $\frac{3abc-3abd}{5fc-5fd} \times \frac{4(a+b)}{a^2-b^2}$.

27. $\frac{3x^2-x}{a} \times \frac{2a}{2x^2-4x}$.

28. $\frac{3x^2}{5x-10} \times \frac{3x-6}{4x^3}$.

29. $\frac{(x-y)^2}{x+y} \times \frac{y}{x(x-y)}$.

30. $\frac{x^4-y^4}{x^2-2xy+y^2} \times \frac{x-y}{x^2+xy}$.

31. $\frac{x(a-x)}{a^2+2ax+x^2} \times \frac{a(a+x)}{a^2-2ax+x^2}$.

32. $\frac{a^6-b^6}{a^4+2a^2b^2+b^4} \times \frac{a^2+b^2}{a^2-ab+b^2} \times \frac{a+b}{a^3-b^3}$.

33. $\dfrac{x^2-(m+n)x+mn}{x^2-(m+p)x+mp}\times\dfrac{x^2-p^2}{x^2-n^2}$.

34. $\dfrac{a^3-b^3}{a^3+b^3}\times\dfrac{a+b}{a-b}\times\left(\dfrac{a^2-ab+b^2}{a^2+ab+b^2}\right)^2$.

35. $\dfrac{a^3+3a^2b+3ab^2+b^3}{a^2-b^2}\times\dfrac{a+b}{a^2+2ab+b^2}$.

36. $\dfrac{2a^3+13a^2x-15ax^2-126x^3}{a^2x+6ax^2-7x^3}\times\dfrac{2a^3+19a^2x+35ax^2}{a^3-a^2x-4ax^2-6x^3}$.

37. $(a+b)\left(\dfrac{1}{a}+\dfrac{1}{b}\right)$.

38. $(2a+3b)\left(\dfrac{b}{a}-\dfrac{a}{b}\right)$.

39. $\left(\dfrac{1}{a}+\dfrac{1}{b}\right)\left(\dfrac{1}{a}-\dfrac{1}{b}\right)$.

40. $\left(\dfrac{3a}{4b}-\dfrac{2c}{3d}\right)\left(\dfrac{4d}{3c}-\dfrac{3b}{2a}\right)$.

41. $\dfrac{x^2+xy}{x^2+y^2}\left(\dfrac{x}{x-y}-\dfrac{y}{x+y}\right)$.

42. $\left(1-\dfrac{a-b}{a+b}\right)\left(2+\dfrac{2b}{a-b}\right)$.

43. $\dfrac{a^2-x^2}{a+b}\times\dfrac{a^2-b^2}{ax+x^2}\times\left(a+\dfrac{ax}{a-x}\right)$.

44. $\left(\dfrac{a}{bc}-\dfrac{b}{ac}-\dfrac{c}{ab}-\dfrac{2}{a}\right)\left(1-\dfrac{2c}{a+b+c}\right)$.

45. $\left(\dfrac{x^2}{a^2}+\dfrac{a^2}{x^2}-\dfrac{x}{a}-\dfrac{a}{x}+1\right)\left(\dfrac{x}{a}-\dfrac{a}{x}\right)$.

46. $\left(\dfrac{x}{a}-\dfrac{a}{x}+\dfrac{y}{b}-\dfrac{b}{y}\right)\left(\dfrac{x}{a}-\dfrac{a}{x}-\dfrac{y}{b}+\dfrac{b}{y}\right)$.

47. $\dfrac{2a^2b^3c}{3x^2y^3}\times\dfrac{a^mb^nc^r}{x^my^n}\times\dfrac{6x^{m-1}y^{n-2}}{a^{m+1}b^{n+2}c^{r+3}}$.

48. $\dfrac{9a^2b^3}{8x^3y^n}\times\dfrac{10a^{n-1}x^{m+2}}{21b^{m+3}y^{m-n}}\times\dfrac{x}{a}$.

49. $\dfrac{3a^3c^n}{7x^3b^n}\times\dfrac{49x^{n-1}b^{n+2}}{9a^{n+5}c^{n+1}}\times\dfrac{a^2x}{c}$.

50. $\dfrac{a^{x-1}b^{x-2}c^{x-3}}{x^{n-1}y^{n+2}z^{n-3}} \times \dfrac{x^{n+1}y^{n-2}z^{n+4}}{a^{x-2}b^{x+1}c^{x-1}} \times \dfrac{a^2c^2}{b^2x^2}$.

51. $\dfrac{a^{m-n}b^{n-p}c^{p-m}}{x^{n-p}y^{p-m}z^{m-n}} \times \dfrac{a^{n-p}b^{p-m}c^{m-n}}{x^{p-1}y^{m-2}z^{n-3}} \times \dfrac{x^ny^pz^m}{a^mb^nc^p}$.

52. $\dfrac{18xy}{25zu} \div \dfrac{6mn}{35pq}$.

53. $\dfrac{7ab}{3mn} \div \dfrac{5pq}{11xy}$.

54. $\dfrac{14a^2b^3c}{39d^2f^5g^6} \div \dfrac{35d^7f^4g^8}{9a^4b^5c^2}$.

55. $\dfrac{25p^4q^5r^6}{49x^4y^5z^6} \div \dfrac{30p^7q^7r^8}{77xy^7z^2}$.

56. $\dfrac{m+n}{m-n} \div \dfrac{n+m}{n-m}$.

57. $\dfrac{3p-3q}{5p+5q} \div \dfrac{9q-9p}{10q+10p}$.

58. $\dfrac{x^2+y^2}{x^2-y^2} \div \dfrac{3x^2+3y^2}{x+y}$.

59. $\dfrac{6ab-6b^2}{a(a+b)} \div \dfrac{2b^2}{a(a^2-b^2)}$.

60. $\dfrac{45(x-y)}{32(z+y)} \div \dfrac{27(x-y)^2}{128b(z+y)^2}$.

61. $\dfrac{a^2-4x^2}{a^2+4ax} \div \dfrac{a^2-2ax}{ax+4x^2}$.

62. $\dfrac{8a^3}{a^3-b^3} \div \dfrac{4a^3}{a^2+ab+b^2}$.

63. $\dfrac{x^3+3x^2y+3xy^2+y^3}{a^3+b^3} \div \dfrac{(x+y)^2}{a^2-ab+b^2}$.

64. $\dfrac{y^2+(a+c)y+ac}{y^2+(b+c)y+bc} \div \dfrac{y^2-a^2}{y^2-b^2}$.

65. $\dfrac{x^2+y^2+2xy-z^2}{z^2-x^2-y^2+2xy} \div \dfrac{x+y+z}{y+z-x}$.

66. $\dfrac{a^2+ab+b^2}{a^3+b^3} \div \dfrac{a^3-b^3}{a^2-ab+b^2}$.

67. $\dfrac{a^2-3a+2}{a^2-6a+9} \div \dfrac{a^2-2a+1}{a^2-5a+6}$.

68. $\dfrac{x^3+3x^2y+3xy^2+y^3}{a^3-b^3} \div \dfrac{(x+y)^2}{a^2+ab+b^2}$.

69. $\dfrac{m^3-y^3}{m^2y^2-m^4} \div \dfrac{m^2+my+y^2}{my^2+y^3}$.

70. $\dfrac{a^2+ab}{a^2+b^2} \div \dfrac{ab\,(a+b)^2}{a^4-b^4}$.

71. $\dfrac{a^4-3a^3x+3a^2x^2-ax^3}{a^3b-b^4} \div \dfrac{a^4-2a^3x+a^2x^2}{a^2b^2+ab^3+b^4}$.

72. $\dfrac{ax-a}{bx+b} \div \dfrac{c-cx}{2b+bx}$.

73. $\dfrac{64m^3-96m^2n+36mn^2}{36m^2n-729n^3} \div \dfrac{48m^2n-27n^3}{8m^3-72m^2n+162mn^2}$.

74. $\dfrac{p^2+3p+9}{p^4-3p^2+9} \div \dfrac{p^3-27}{p^6+27}$.

Ex. 14.

Complex Fractions.

Simplify the following expressions:

1. $\dfrac{\frac{x}{m}+\frac{y}{m}}{\frac{z}{m}}$.

2. $\dfrac{\frac{m}{x}-\frac{c}{x}}{\frac{a}{x}+\frac{d}{x}}$.

3. $\dfrac{\frac{2}{5}mn-\frac{1}{5}mp}{np}$.

4. $\dfrac{\frac{2}{3}mn-\frac{5}{8}mn}{\frac{2}{3}m}$.

5. $\dfrac{x-\frac{2}{3}y}{z-\frac{1}{3}u}$.

6. $\dfrac{x-\frac{y}{z}}{z-\frac{y}{x}}$.

7. $\dfrac{\frac{ab}{c}-3d}{3c-\frac{ab}{d}}$.

8. $\dfrac{1+\frac{1}{a-1}}{1-\frac{1}{a+1}}$.

9. $\dfrac{1+\frac{y}{x-y}}{1-\frac{y}{x+y}}$.

10. $\dfrac{m+\frac{mn}{m-n}}{m-\frac{mn}{m+n}}$.

11. $\dfrac{\dfrac{x+y}{x-y}-\dfrac{x-y}{x+y}}{\dfrac{x-y}{x+y}+\dfrac{x+y}{x-y}}$

12. $\dfrac{\dfrac{m}{n}+\dfrac{p}{q}}{\dfrac{o}{n}-\dfrac{t}{q}}$

13. $\dfrac{x+y+\dfrac{y^2}{x}}{x+y+\dfrac{x^2}{y}}$

14. $\dfrac{x+\dfrac{y-x}{1+xy}}{1-\dfrac{(y-x)x}{1+xy}}$

15. $\dfrac{a-1+\dfrac{6}{a-6}}{a-2+\dfrac{3}{a-6}}$

16. $\dfrac{\dfrac{2m+n}{m+n}-1}{1-\dfrac{n}{m+n}}$

17. $\dfrac{\dfrac{x^3+y^3}{x^2-y^2}}{\dfrac{x^3-xy+y^3}{x-y}}$

18. $\dfrac{3}{x+1}-\dfrac{2x-1}{x^2+\dfrac{x}{2}-\dfrac{1}{2}}$

19. $\dfrac{y-a}{y-\dfrac{(y-b)(y-c)}{y+a}}$

20. $1-\dfrac{1}{1+\dfrac{1}{a}}$

21. $1+\dfrac{a}{1+a+\dfrac{2a^2}{1-a}}$

22. $\dfrac{1}{1-\dfrac{1}{1+\dfrac{1}{a}}}$

23. $\dfrac{1}{1+\dfrac{a}{1+a+\dfrac{2a^2}{1-a}}}$

24. $\dfrac{1}{a+\dfrac{1}{1+\dfrac{a+1}{3-a}}}$

25. $\dfrac{a}{b+\dfrac{c}{d+\dfrac{e}{f}}}$

26. $\dfrac{x+\dfrac{1}{y}}{x+\dfrac{1}{y+\dfrac{1}{z}}}-\dfrac{1}{y(xyz+x+z)}$.

27. $\dfrac{3abc}{bc+ca-ab}-\dfrac{\dfrac{a-1}{a}+\dfrac{b-1}{b}+\dfrac{c-1}{c}}{\dfrac{1}{a}+\dfrac{1}{b}-\dfrac{1}{c}}$.

28. $\left(\dfrac{m+n}{m-n}+\dfrac{m^2+n^2}{m^2-n^2}\right)\div\left(\dfrac{m-n}{m+n}-\dfrac{m^3-n^3}{m^3+n^3}\right)$.

29. $\left(\dfrac{x-y}{x+y}-\dfrac{x^3-y^3}{x^3+y^3}\right)\div\left(\dfrac{x+y}{x-y}+\dfrac{x^2+y^2}{x^2-y^2}\right)$.

30. $\left(\dfrac{a^2+b^2}{a^2-b^2}-\dfrac{a^2-b^2}{a^2+b^2}\right)\div\left(\dfrac{a+b}{a-b}-\dfrac{a-b}{a+b}\right)$.

31. $\left(\dfrac{m+n}{m-n}+\dfrac{m-n}{m+n}\right)\div\left(\dfrac{m^2+n^2}{m^2-n^2}-\dfrac{m^2-n^2}{m^2+n^2}\right)$.

32. $\dfrac{\dfrac{a^2+b^2}{b}-a}{\dfrac{1}{b}-\dfrac{1}{a}}\times\dfrac{a^2-b^2}{a^3+b^3}$.

33. $\dfrac{x}{x-y}-\dfrac{x}{x+y}-\dfrac{\dfrac{x+y}{x-y}-\dfrac{x-y}{x+y}}{\dfrac{x+y}{x-y}+\dfrac{x-y}{x+y}}$.

34. $\dfrac{\dfrac{1}{a}+\dfrac{1}{b+c}}{\dfrac{1}{a}-\dfrac{1}{b+c}}\left(1+\dfrac{b^2+c^2-a^2}{2bc}\right)$.

35. $\left(\dfrac{9m^2-3n^2}{4mn}-\dfrac{m-4n}{5n}\right)\div\left(\dfrac{2m+n}{3m}-\dfrac{5n^2-3m^2}{16m^2}\right)$.

36. $\left(\dfrac{10xy-3y^2+10x-3y}{15y^2+10xy^2+30y+20xy} \div \dfrac{10x-3y}{45y+30xy}\right)+\dfrac{3}{y+2}$.

37. $\left(\dfrac{x^3-y^3}{x^2-y^2}-\dfrac{10x^2-13xy-3y^2}{10x^2-3xy-y^2}-\dfrac{2x^3+xy^2+xy+y^2}{2x^2+xy-y^2}\right)$
$\div \dfrac{xy-y^2-2x+2y}{2x-y}$.

38. $\dfrac{\left[\dfrac{(a+b)^2}{4ab}-1\right]\left[\dfrac{(a-b)^2}{4ab}+1\right]}{(a+b)^3-3a^2b-3ab^2} \times \dfrac{[(a+b)^2-ab][(a-b)^2+ab]}{(a-b)^3+3ab(a-b)}$.

39. $\dfrac{\dfrac{x^3-y^3}{x^2+y^2} \times \dfrac{x^2-y^2}{x^3+y^3}\left(\dfrac{1}{x^2}+\dfrac{1}{y^2}\right) \div \left(\dfrac{1}{y}-\dfrac{1}{x}\right)}{\dfrac{(x+y)^2-xy}{(x-y)^2+xy}}$.

40. $\dfrac{\left[\dfrac{(m+n)^3}{3mn}-m-n\right]\left[\dfrac{(m-n)^3}{3mn}+m-n\right]}{\left[\dfrac{(m+n)^2}{4mn}-1\right][(m+n)^2-mn]}$
$\times \dfrac{(n-m)^3-(n-m)(m^2+mn+n^2)}{(m+n)^3-3mn(m+n)}$.

41. $\dfrac{\dfrac{1+x}{1-x}+\dfrac{4x}{1+x^2}+\dfrac{8x}{1-x^4}-\dfrac{1-x}{1+x}}{\dfrac{1+x^2}{1-x^2}+\dfrac{4x^2}{1+x^4}-\dfrac{1-x^2}{1+x^2}}$.

42. $\dfrac{\dfrac{1}{1+x}}{1-\dfrac{1}{1+x}}+\dfrac{\dfrac{1}{1+x}}{\dfrac{x}{1-x}}+\dfrac{\dfrac{1}{1-x}}{\dfrac{x}{1+x}}$.

43. $\left(\dfrac{1}{a+\dfrac{1}{b+\dfrac{1}{c}}} \div \dfrac{1}{a+\dfrac{1}{b}}\right)-\dfrac{1}{b(abc+a+c)}$.

Ex. 15.

THE SYMBOLS $\frac{0}{m}$, $\frac{m}{0}$, $\frac{0}{0}$.

1. What will be the quotient when the dividend is equal to 0, and the divisor is any assigned quantity?

2. What will be the quotient $\frac{1}{b}$ when b is greater than any assignable quantity and increases without limit?

3. What would $\frac{n}{k}$ become when k decreases and approaches indefinitely to 0?

4. What is the value of the expression $\frac{4x-3}{24x-18}$ if $x=\frac{3}{4}$?

5. What is the value of $\frac{3x}{7x}$ if $x=0$?

6. What will the fraction $\frac{3x}{7x}$ become if x has any other value than 0?

The symbol $\doteq$ is used for the word *approaches*.

Find the *limits* of the values of the following fractions:

7. $\frac{x^2-8x+15}{x^2-10x+21}$; if $x \doteq 3$.

8. $\frac{a^2-a-2}{a^2-4}$; if $a \doteq 2$.

9. $\frac{x^2-5xy+4y^2}{x^2+xy-2y^2}$; if $x \doteq y$.

10. $\frac{16-a^4}{2-a}$; if $a \doteq 2$.

11. $\frac{x^2-y^2}{2x^2-xy-y^2}$; if $x \doteq y$.

12. $\frac{m^2-2mn+n^2}{m^2-n^2}$; if $m \doteq n$.

13. $\frac{p^2-pq-2q^2}{p^2-4pq+4q^2}$; if $p \doteq 2q$.

14. $\frac{m^4-5m^2n^2+4n^4}{m^4-3m^2n^2-4n^4}$; if $m \doteq 2n$.

15. $\frac{a^2-6ab+9b^2}{2a^2-ab-15b^2}$; if $a=3b$. 16. $\frac{x^3-a^3}{x^2-a^2}$; if $x=a$.

17. $\frac{x^3-15x^2+74x-120}{x^3-12x^2+41x-30}$; 1st, when $x=5$; 2d, when $x=6$.

18. $\frac{m^2+2mn+n^2-4}{m+n-2}$; if $m=1$, and $n=1$.

19. $\frac{x^3-4}{x^2-3x+2}$; if $x=2$. 21. $\frac{2x^2-3x-9}{3x^2-10x+3}$; if $x=3$.

20. $\frac{x^3+2x-3}{x^3+3x-4}$; if $x=1$. 22. $\frac{x^3-2x+1}{x^3-5x+4}$; if $x=1$.

23. $\frac{x^3-6x^2+6x-5}{x^3-2x^2-19x+20}$; if $x=5$.

24. $\frac{2x^2+5x-3}{x^2+x-6}$; if $x=-3$.

25. $\frac{3a^3-3\frac{1}{2}a^2+a}{2a^3-6a^2+\frac{5}{2}a}$; if $a=\frac{1}{2}$.

26. $\frac{5a^2-9a-2}{10a^2-13a-3}$; if $a=-\frac{1}{5}$.

27. $\frac{2x^3-5ax^2+3a^2x}{3x^3-8ax^2+5a^2x}$; if $x=a$.

28. $\frac{2}{x^2-1}-\frac{1}{x-1}$; if $x=1$.

29. $\frac{x^3+2x^2-14x-3}{x^2-x-6}$; if $x=3$.

30. $\frac{4x^6-6ax^2+2ax^5-3a^2x}{6x^5-4ax^3+3ax^4-2a^2x}$; if $x=-\frac{a}{2}$

31. $\frac{a(x^2+c^2)-2acx}{b(x^2+c^2)-2bcx}$; if $x=c$.

32. $\frac{x^3+4x^2+x-6}{x^3+7x^2+14x+8}$; if $x=-2$.

Ex. 16.

Various Exercises in Fractions.

Find the value of the following expressions:

1. $\frac{m-x}{n-x}$; if $x=\frac{mn}{m+n}$.

2. $\frac{x-m}{n}-\frac{x-n}{m}$; if $x=\frac{m^2}{m-n}$.

3. $\frac{x}{a}+\frac{x}{b-a}-\frac{a}{a+b}$; if $x=\frac{a^2(b-a)}{b(b+a)}$.

4. $\frac{x+2a}{2b-x}+\frac{x-2a}{2b+x}-\frac{4ab}{4b^2-x^2}$; if $x=\frac{ab}{a+b}$.

5. $\frac{x+y-1}{x-y+1}$; if $x=\frac{a+1}{ab+1}$ and $y=\frac{a(b+1)}{ab+1}$.

6. $\left(\frac{x-a}{x-b}\right)^2-\frac{x-2a+b}{x+a-2b}$; if $x=\frac{a+b}{2}$.

7. If $y=\frac{1-z^2}{1+z^2}$, $z=\frac{1-x}{1+x}$, express the value of y in terms of x.

8. What will the expression $(a^2-x)a+bx$ equal, when $x=a^2+ab+b^2$?

9. What will $\frac{x^2+2x-8}{x^2-x-4}$ equal, when $x=0$?

10. What will $\frac{ab^2}{mb^2-b}$ equal, when $b=\infty$?

Simplify the following expressions:

11. $\dfrac{(1-xy)(1+xy)-(x-y)(x+y)}{(1+xy)^2+(x-y)^2}$.

12. $\dfrac{a^2-bc}{(a+b)(a+c)}+\dfrac{b^2-ac}{(b+a)(b+c)}+\dfrac{c^2-ab}{(c+a)(c+b)}$.

13. $\dfrac{a^2}{(a-b)(a-c)}+\dfrac{b^2}{(b-a)(b-c)}+\dfrac{c^2}{(c-a)(c-b)}$.

The first member of the following equalities being given, through what transformations can it be changed to the form indicated by the second member?

14. $(2x^2-1)\left[\dfrac{x}{1-x}+\dfrac{1+x}{x}\right]=\dfrac{x}{1-x}-\dfrac{1+x}{x}$.

15. $\dfrac{a^6+2a^3b^3+b^6}{a^2+2ab+b^2}=(a^2-ab+b^2)^2$.

16. $\dfrac{a-c}{(a-b)(x-a)}-\dfrac{b-c}{(a-b)(x-b)}=\dfrac{x-c}{(x-a)(x-b)}$.

17. $\dfrac{1}{x(x-a)(x-b)}=\dfrac{1}{abx}+\dfrac{1}{a(a-b)(x-a)}-\dfrac{1}{b(a-b)(x-b)}$.

18. $\dfrac{x}{1-x^2}+\dfrac{y}{1-y^2}+\dfrac{z}{1-z^2}=\dfrac{4xyz}{(1-x^2)(1-y^2)(1-z^2)}$, when $xy+yz+xz=1$.

19. Change $1-\dfrac{b^2+c^2-a^2}{2bc}$ into a product of two factors.

20. What is, in the development of $\dfrac{1}{1-x}$, the nth term, and what is the fraction that remains?

21. When, in the preceding example, we have $x < \frac{1}{2}$, we also have $\frac{x^n}{1-x} < x^{n-1}$. If, then, $2x^{n-1}$ is small enough not to affect the degree of exactness proposed, we can stop at the $n-1$th term, and write:

$$\frac{1}{1-x} = 1 + x + x^2 \cdots\cdots + x^{n-2}.$$

Application: change $\frac{1}{0.97}$ into a decimal fraction, exact to the seventh figure.

22. When $\frac{1}{x}$ is small, since $\frac{a}{x+b} = \frac{a}{x} - \frac{ab}{x^2} + \frac{ab^2}{x^3} \cdots\cdots$ a quantity in a decimal fraction can be developed by approaching rapidly with the aid of this formula to a given degree of approximation. Ex.: reduce $\frac{8}{107}$ to a decimal fraction, exact to the units of the sixth decimal place.

23. Reduce $\frac{8}{104}$ to a decimal fraction, exact to the ten millionths.

24. Change $\frac{5}{0.98}$ to a decimal fraction, exact to the eighth figure.

25. Change $\frac{7}{0.995}$ to a decimal fraction, exact to the twelfth figure.

26. Change $\frac{8}{1001}$ to a decimal fraction, exact to the thirteenth figure.

27. Reduce to a decimal fraction: $\dfrac{1+\dfrac{2}{3\frac{4}{5}}}{1-\dfrac{2}{3\frac{4}{5}}} \div \left(1+\dfrac{4}{9-\dfrac{3}{1-\frac{3}{5}}}\right)$.

Ex. 17.

Simple Numerical Equations with One Unknown Quantity.

Solve:

1. $30x - 33 - 36x - 39 = 39 + 36x + 33 - 30x.$
2. $7x - 9 - 18x + 7 = 18x + 9 - 7x - 7.$
3. $15x + 372 + 9x = -618 - 5x - 53 + 9x.$
4. $21x - 42 + 5x - 15 = 11 - 68 + 4x.$
5. $4x - 60 + 3x - 6x = 11 - 77 + 7x.$
6. $28x - 21 + 42 - 28x = 22 - 4x.$
7. $4(5x - 3) = 104 - 9x.$
8. $8(10 - x) = 5(x + 3).$
9. $7(15 - 3x) = 6(x + 4).$
10. $8(9 - 2x) = 5(3x + 2).$
11. $8(37 - 5x) = 4(3x - 17).$
12. $9(23 - 5y) = 8(5y - 6).$
13. $9(13 - x) - 4x = 5(21 - 2x) + 9x.$
14. $6(x - 5) + 2x = 8x - 2(x + 10).$
15. $199 + 15x - (2x - 5) = 17(x + 17) - 13x - 22.$
16. $118x - 13(5 + 11x) + 15(3x - 3) = 18x - 7(x - 5) - 119.$
17. $8(3x - 2) - 7x - 5(12 - 3x) + 28 = 8(3x + 2) - 32.$
18. $7(3x - 6) + 5(x - 3) + 4(17 - x) = 11.$
19. $2(16 - x) + 3(5x - 4) = 12(3 + x) - 2(12 - x).$
20. $3(3x - 2) - 5(18 - 5x) = 13(8x + 5) + 5(9x + 6).$
21. $28x + 49(11 - x) = 42x + 18(20 - x) + 71 - 3x + 33(3x - 1).$
22. $364 - 15(15x - 7) - 3(13 - 2x) + 8 = 6(x - 2).$
23. $7(18 - 3x) = 111 - 3(20 - 5x) + 39x.$
24. $22 - 2(18 - x) - 3(15 - x) + 9 - 8(x - 10) = 0.$
25. $5(x - 3) - 18(5x - 3) + 27 = 32(9x - 5) - 9x + 670.$
26. $\frac{x}{2} - \frac{4x}{9} = 5.$
27. $\frac{3x}{4} - \frac{2x}{7} = 13.$

28. $\frac{z}{2}+\frac{z}{3}-\frac{z}{4}=7.$

29. $\frac{x}{5}+\frac{3x}{7}-\frac{x}{2}=9.$

30. $3y-\frac{5}{6}y+\frac{1}{2}y=32.$

31. $2z+\frac{3}{4}z-\frac{5}{7}z=57.$

32. $13x-\frac{8x}{9}+\frac{7x}{2}=15x+22.$

33. $\frac{2c}{5}-\frac{3c}{2}=\frac{c}{2}-32.$

34. $3x+\frac{1}{2}(5x-3)=20\frac{1}{2}.$

35. $\frac{5}{6}(3x-7)=\frac{3}{4}x+4\frac{2}{3}.$

36. $2\frac{1}{2}(3x-11)=3\frac{1}{3}x+18\frac{1}{3}.$

37. $19x+\frac{1}{2}(7x-2)=4x+17\frac{1}{2}.$

38. $x-1\frac{1}{2}(5+\frac{3}{4}x)+\frac{1}{3}(4-x)-\frac{1}{3}(3x-16)=0.$

39. $7+18\frac{3}{8}x+49(\frac{3}{8}x-2\frac{1}{2})=21\left(\frac{15}{2}+\frac{3x}{4}\right).$

40. $\frac{x-2}{3x-5}=\frac{6}{19}.$

41. $\frac{5x-3}{7x+9}=\frac{7}{23}.$

42. $\frac{3(5x-3)}{2(4x+3)}=\frac{6}{5}.$

43. $\frac{\frac{3}{4}x-1}{11-\frac{2}{3}x}=\frac{8}{3}.$

44. $\frac{\frac{1}{2}(3x-1)}{\frac{1}{3}(4-x)}=\frac{3}{2}.$

45. $\frac{0.25x-1.5}{0.15(x-5)}=\frac{2}{1.5}.$

46. $\frac{0.125(0.1x+0.5)}{0.05(7x-30)}=0.5.$

47. $\frac{2(x+1.5)}{5(0.8x-1)}=\frac{15}{19}.$

48. $\frac{3x+9}{4}=\frac{5x+16}{7}.$

49. $\frac{5-x}{7}=\frac{11-3x}{23}.$

50. $\frac{5x-6}{5}-\frac{3x}{13}=\frac{x-4}{9}.$

51. $\frac{12-3x}{4}-\frac{3x-11}{3}=1.$

52. $\frac{2(7x-10)}{3}-20=\frac{50-x}{2}.$

53. $\frac{x-5}{4}-\frac{284-x}{5}-6x=0.$

54. $\frac{2x+1}{3}+\frac{3x+1}{4}=28-\frac{5x-2}{7}$.

55. $\frac{5x-6}{7}-\frac{3x-4}{9}-\frac{4x}{11}=0$.

56. $\frac{x+1}{2}+\frac{3x-4}{5}+\frac{1}{8}=\frac{6x+7}{8}$.

57. $\frac{5x-11}{4}-\frac{x-1}{10}=\frac{11x-1}{12}$.

58. $\frac{8-x}{6}+\frac{3x-5}{3}=\frac{x+6}{2}-\frac{x}{3}$.

59. $\frac{x+3}{2}-\frac{1}{8}(x-2)=\frac{1}{12}(3x-5)+\frac{1}{4}$.

60. $11-\left(\frac{3x-1}{4}+\frac{2x+1}{3}\right)=10-\left(\frac{2x-5}{3}+\frac{7x-1}{8}\right)$.

61. $\frac{x-1}{3}+\frac{16x-3}{20}-\frac{7x-6}{8}=\frac{x+2}{2}+\frac{3(x-3)}{10}$.

62. $\frac{x-3}{4}-\frac{2x-5}{6}-\frac{41}{60}-\frac{3x-8}{5}+\frac{5x+6}{15}=0$.

63. $\frac{3x+5}{8}-\frac{5x-4}{9}+\frac{7x-5}{6}=\frac{9x-53}{5}-\frac{3x}{17}$.

64. $\frac{4x-2}{9}-\frac{3x-5}{8}+\frac{7x-1}{5}=\frac{2(5x-13)}{7}+\frac{2x}{23}$.

65. $\frac{7x-4}{9}+\frac{3x-1}{5}-\frac{5(x-1)}{6}=\frac{3(3x-1)}{20}-\frac{9x}{7}$.

66. $15x-\frac{5(2x+5)}{7}=80-2\frac{1}{2}(7x+19)-\frac{5}{8}(2x+1)$.

67. $6x-\frac{27x-1}{4}-\frac{2(4x-1)}{5}=\frac{9x-5}{4}-\frac{11x-2}{3}+22$.

68. $\frac{6(x+3)}{13} - 3\frac{1}{6} - \frac{11-3x}{36} = 5x - 48 - \frac{13-x}{12} - \frac{21-2x}{18}$.

69. $\frac{3x+4}{7} - \frac{9x+44}{5} + \frac{3(3x+10)}{4} = \frac{5x+12}{3}$.

70. $\frac{102x-1}{4} - \frac{8(x-1)}{5} = \frac{18x-1}{4} - \frac{2(11x-1)}{3} + 21$.

71. $\frac{10x+11}{6} - \frac{14x-13}{3} - 4 = \frac{7-6x}{4}$.

72. $\frac{11x-13}{25} + \frac{19x+3}{7} - \frac{5x-25\frac{1}{2}}{4} = 28\frac{1}{7} - \frac{17x+4}{21}$.

73. $\frac{4x-21}{7} + \frac{47}{6} + \frac{7x-28}{3} = \frac{4x+15}{4} - \frac{9-7x}{8} + \frac{1}{12}$.

74. $\frac{x-3\frac{3}{4}}{2} - \frac{5-4x}{7} = x + 21\frac{6}{7} - \frac{3x-\frac{1}{2}(3-2x)}{6}$.

75. $\frac{4x}{15} - \frac{9}{10} - \frac{16x-81}{24} = \frac{2x-3}{15} - \frac{4x-9}{20} + \frac{9}{40}$.

76. $\frac{19(13.4-2.5x)}{4} + 7.1 = 39.7625 - 9(0.45+8x)$.

77. $\frac{1+2x}{3} - \frac{5-3x}{3} + 4 - \frac{8x}{5} = \frac{x}{1.5} + \frac{x}{1.05}$.

78. $\frac{3x}{8} - \frac{0.25x}{0.4} + 3.642 - \frac{x}{0.8} + 0.01 = 0$.

79. $\frac{5(3x-2)}{4} + \frac{3x}{2} - 23\frac{5}{6} = \frac{x - \frac{4x-9}{3}}{6} + 5$.

80. $\frac{x - \frac{2(x-18)}{9}}{8} - \frac{x-18}{6} = x + 9 - \frac{5x - \frac{2(x+10)}{23}}{4}$.

81. $\dfrac{2\frac{1}{3}x-2}{4}-\dfrac{\frac{10x-1}{2}}{3}=\dfrac{\frac{x-8}{4}}{5}-3\frac{1}{3}$.

82. $\dfrac{3}{y-4}+\dfrac{5}{2(y-4)}+\dfrac{9}{2(y-4)}=\dfrac{1}{2}$.

83. $\dfrac{2}{z-1}-\dfrac{5}{2z-2}=\dfrac{8}{3z-3}-\dfrac{z}{z-1}+\dfrac{5}{18}$.

84. $\dfrac{5}{m+3}+\dfrac{3}{2(m+3)}+\dfrac{42}{12(m+3)}=0.5$.

85. $\dfrac{2x-3}{2x-4}-6=\dfrac{x+5}{3x-6}-\dfrac{11}{2}$.

86. $\dfrac{2x-13}{2x-16}+\dfrac{2(x-6)}{x-8}=\dfrac{7}{8}+\dfrac{2(5x-39)}{3x-24}$.

87. $\dfrac{10-7x}{6-7x}=\dfrac{5x-4}{5x}$.

88. $\dfrac{29-10y}{9-5y}=\dfrac{5+36y}{18y}$.

89. $\dfrac{5+8z}{3+2z}=\dfrac{45-8z}{13-2z}$.

90. $\dfrac{47+6x}{11+2x}=\dfrac{2(8+3x)}{3+2x}$.

91. $\dfrac{5u-16}{u-5}=\dfrac{5u-3}{u+5}$.

92. $\dfrac{5x-1}{2x+3}=\dfrac{5x-3}{2x-3}$.

93. $\dfrac{3z+7}{4z+5}=\dfrac{3z+5}{4z+3}$.

94. $\dfrac{x}{3}-\dfrac{x^2-5x}{3x-7}=\dfrac{2}{3}$.

95. $\dfrac{4y+3}{9}-\dfrac{8y+19}{18}=-\dfrac{7y-29}{5y-12}$.

96. $\dfrac{11m-13}{14}-\dfrac{22m-75}{28}=\dfrac{13m+7}{2(3m+7)}$.

97. $\dfrac{16a+7}{24}-\dfrac{2a+1}{3}=\dfrac{a-16}{3(3a-59)}$.

98. $\frac{3x+2}{x-1}+\frac{2x-4}{x+2}=5.$

99. $\frac{3x-1}{2x-1}-\frac{2(2x-1)}{3x-2}=\frac{1}{6}.$

100. $\frac{a^2-a+1}{a-1}+\frac{a^2+a+1}{a+1}=2a.$

101. $\frac{4x+17}{x+3}+\frac{3x-10}{x-4}=7.$

102. $\frac{4}{1+x}+\frac{x+1}{1-x}=\frac{x^2-3}{1-x^2}.$

103. $\frac{5-2b}{b-1}-\frac{2-7b}{b+1}=\frac{5b^2+4}{b^2-1}.$

104. $\frac{6}{x+2}-\frac{x+2}{x-2}+\frac{x^2}{x^2-4}=0.$

105. $\frac{9y+5}{6(y-1)}+\frac{3y^2-51y-71}{18(y^2-1)}=\frac{15y-7}{9(y+1)}.$

106. $\frac{2u+1}{2u-1}-\frac{8}{4u^2-1}=\frac{2u-1}{2u+1}.$

107. $\frac{1+3m}{5+7m}-\frac{9-11m}{5-7m}=\frac{14(2m-3)^2}{25-49m^2}.$

108. $\frac{2y+1}{3y-3}=\frac{7y-1}{6y+6}-\frac{2y^2-3y-45}{4y^2-4}.$

109. $\frac{4u+3}{15u-35}-\frac{11u-5}{9u+21}=\frac{375u-86u^2-35}{10(9u^2-49)}.$

110. $\frac{2z+3}{5z-7}-\frac{3z+4}{7z-5}=\frac{25z-3(2z^2-73)}{6(35z^2-74z+35)}$

Ex. 18.

LITERAL EQUATIONS WITH ONE UNKNOWN QUANTITY.

Solve:

1. $\frac{m}{px}+n=\frac{k}{px}+q.$

2. $\frac{a}{bz}+c=\frac{d}{gz}+3.$

3. $\frac{c+d}{x}-a=\frac{c-d}{x}+b.$

4. $\frac{m+n}{px}-\frac{c-d}{qx}=1.$

5. $\frac{m+1}{m+x}=\frac{n}{m-x}.$

6. $\frac{p}{m(n+x)}=\frac{x}{n+x}.$

7. $\frac{z}{a+az}=\frac{b}{c+cz}.$

8. $\frac{c+d}{ab+by}=\frac{m-y}{an+ny}.$

9. $\frac{1+z}{1-z}=\frac{1}{m}.$

10. $\frac{m+ny}{p+qy}=b.$

11. $\frac{my+n}{my-n}=\frac{a}{b}.$

12. $\frac{m+z}{n+3z}=\frac{3}{4}.$

13. $\frac{p(q+x)}{p-x}=q.$

14. $\frac{m}{m-x}=\frac{n}{n-x}.$

15. $\frac{z+2}{z-2}=\frac{m+n}{m-n}.$

16. $\frac{m+n}{p+x}=\frac{m-n}{p-x}.$

17. $\frac{a+bx}{a+b}=\frac{c+dx}{c+d}.$

18. $\frac{a+bx}{a+b}=\frac{c-dx}{c-d}.$

19. $\frac{m-z}{n-z}=\frac{m+z}{n+z}.$

20. $\frac{6x+a}{4x+b}-\frac{3x-b}{2x-a}=0.$

21. $\frac{x}{a}+\frac{x}{b}+\frac{x}{c}=d.$

22. $\frac{ax}{b}-\frac{cx}{d}+\frac{fx}{g}=h.$

23. $\frac{ax-b}{c}-\frac{bx+c}{a}=abc.$

24. $\frac{a-x}{b}-\frac{b-x}{a}+\frac{c-x}{c}=a^2-b.$

25. $\frac{x+a}{b}-x=b-\frac{x-b}{c}+\frac{c-bx}{b}.$

26. $\frac{3-x}{a}-\frac{5+2x}{b}=c-\frac{dx}{3}+\frac{(2b-7a)x+5b}{3ab}.$

27. $\frac{20p-qx}{5p}+\frac{9n-px}{3n}+\frac{6m-nx}{2m}=10.$

28. $\frac{3b(x-a)}{5a}+\frac{x-b^2}{15b}=-\frac{(4a+cx)b}{6a}.$

29. $\frac{mx}{n}-\frac{n-x}{2p}+\frac{m(n-x)}{3q}=m.$

30. $\frac{3ax-2b}{3b}-\frac{ax-a}{2b}=\frac{ax}{b}-\frac{2}{3}.$

31. $\frac{a^2-x}{b}-\frac{ab-x}{c}+\frac{c^2-x}{a}+\frac{a^2b+cx}{ab}=\frac{a^2}{b}+\frac{c^2}{a}$

32. $\frac{b+c}{x}+a=\frac{a}{x}-b+\frac{a(a-1)-b(b-1)+c}{x}.$

33. $\frac{(bd+1)x}{bd}+\frac{3x}{bc^2}-\frac{0.6x}{cd^2}-\frac{x}{bd}=x+0.84.$

34. $\frac{b-2x}{3a}-\frac{5a^2}{2b^2}=\frac{x}{a}-2+\frac{a(x-a)}{b^2}.$

35. $\frac{3a^2}{4b}+2x-\frac{(a-b)^2}{2a}=\frac{3bx}{5a}+b.$

36. $\frac{m}{n}:\frac{p}{q}=\frac{q}{p}:x.$ 37. $\frac{m}{n}:\frac{p}{q}=a:\frac{1}{x}.$

38. $\dfrac{mx+n}{p} : \dfrac{ax-b}{c} = d : f.$

39. $\dfrac{a+1}{x} : \dfrac{b}{x} = (a+x) : (b+x).$

40. $\dfrac{\dfrac{m+n}{x}}{\dfrac{1}{m}} = \dfrac{p}{q}.$

41. $\dfrac{\dfrac{mx-n}{x}}{\dfrac{m}{n}} = \dfrac{\dfrac{mx+n}{x}}{\dfrac{n}{m}}.$

42. $\dfrac{\dfrac{cx+d}{m}}{\dfrac{cx}{d}} = \dfrac{2d}{m}.$

43. $\dfrac{\dfrac{a+x-3}{3}}{\dfrac{a-x+3}{3}} = \dfrac{b}{c}.$

44. $p - \dfrac{x-np}{m} = \dfrac{x-mp}{n} - \dfrac{x-mn}{p} - p.$

45. $\dfrac{x}{mn} + \dfrac{x}{mp} + \dfrac{x}{np} - 1 = mnp - (m+n+p)x.$

46. $\dfrac{5ax}{a-b} - 3a = 8x.$

47. $\dfrac{5x}{3a+b} - 2 = \dfrac{8b}{5a}.$

48. $\dfrac{3a+x}{5a-b} = \dfrac{4x-b}{3a+2b} - 1.$

49. $\dfrac{x}{a-b} - \dfrac{5a}{a+b} = \dfrac{2bx}{a^2-b^2}.$

50. $\dfrac{z-2a}{z+3a} = 3 - \dfrac{2z^2-13a^2}{z^2-9a^2}.$

51. $\dfrac{c^2-3dy}{c^2+3dy} + \dfrac{d^2+2cy}{d^2-2cy} = 2.$

52. $\dfrac{1}{n} + \dfrac{n}{x+n} = \dfrac{x+n}{nx}.$

53. $\dfrac{1}{ab-ax} + \dfrac{1}{bc-bx} = \dfrac{1}{ac-ax}.$

54. $\dfrac{x}{a} + \dfrac{x}{b-a} = \dfrac{a}{b+a}.$

55. $\dfrac{z+a}{z+b} = \left(\dfrac{2z+a+c}{2z+b+c}\right)^2.$

56. $\dfrac{m}{z-m} - \dfrac{n}{z-n} = \dfrac{m-n}{z-p}.$

57. $\dfrac{m}{z+m} + \dfrac{n}{z+n} = \dfrac{m+n}{z+p}.$

58. $\frac{1}{y-m}-\frac{1}{y-n}=\frac{m-n}{y^2-mn}$. 60. $\frac{x-c}{c-d}-\frac{x+c}{c+d}=\frac{2cx}{c^2-d^2}$.

59. $\frac{mz-a-b}{nz-c-d}=\frac{mz-a-c}{nz-b-d}$. 61. $\frac{y-a}{y-b}=\left(\frac{2y-a}{2y-b}\right)^2$.

62. $\frac{z}{a-b}-\frac{5a}{a+b}=\frac{2bz}{a^2-b^2}$.

63. $\frac{6a+5b}{6a}-\frac{4bx}{3a^2}=1-\frac{bx}{a^2+ab}$.

64. $\frac{7z}{9m^2}-\frac{3}{2m}=\frac{9n+14z}{18m^2-24mn}$.

65. $\frac{3y}{a+3b}+\frac{18by}{a^2-9b^2}=1$. 66. $\frac{y}{3a+y}+\frac{y}{3a-y}=\frac{a^2}{9a^2-y^2}$.

67. $\frac{m(2x-m)}{m+2n}+\frac{n(2x-n)}{n+2m}=x$.

68. $\frac{z}{p+q}+pqz=p+q+\frac{1}{pq}$.

69. $p^2=\frac{p^3-q^3}{p-q}-\frac{pq(p+q)}{x}$. 70. $q^2-\frac{p^3-q^3}{p-q}=-\frac{pq(p+q)}{x}$.

71. $\frac{m+nx}{p+qx}=\frac{qrx^2}{p(p+qx)}+\frac{s-rx}{p}$.

72. $\frac{3def}{d+e}+\frac{d^2e^2}{(d+e)^2}+\frac{(2d+e)e^2x}{d(d+e)^2}=3fx+\frac{ex}{d}$.

73. $\frac{k}{z}+\frac{z}{k}+\frac{k(z-k)}{z(z+k)}-\frac{z(z+k)}{k(z-k)}=\frac{kz}{k^2-z^2}-2$.

74. $\frac{f+3y}{4f^2+6fg}-\frac{2y-f}{6fg-9g^2}=\frac{2f+y}{4f^2-9g^2}$.

75. $\frac{3bx}{2a^2}-\frac{x-b}{a+b}-\frac{bx-a^2}{a^2-b^2}+\frac{x}{4a}=0$.

Ex. 19.

PROBLEMS WITH ONE UNKNOWN QUANTITY.

1. If the population of a city increases 5 per cent annually, and is now 194,481, what was it 4 years ago?
2. In order to dig a ditch A and B work together 40 days, B and C 48 days, A and C 36 days. How many days would it take for each of them to do the work alone, and how many if they all work together?
3. The two arms of the beam of a false balance are respectively 6 inches and 6.2 inches. It is necessary to put in the pan suspended to the longer arm weights of $2\frac{1}{2}$ pounds in order to make equilibrium with a body placed in the other pan. What is the exact weight of the body?
4. A bamboo 10 feet high was bruised at a certain height. When the upper part was bent and touched the ground the top of the bamboo was 3 feet from the base. At what height was it bruised?
5. The fore and hind wheels of a carriage are respectively 3.25 yards and 3.8 yards in circumference. What distance would the carriage have passed over when one of the fore wheels makes 143 revolutions more than one of the hind wheels?
6. A train which travels 32 miles an hour is $\frac{3}{4}$ of an hour in advance of a second train which travels 42 miles an hour. In how long a time will the last overtake the first?
7. A train travelling b miles per hour is m hours in advance of a second train which travels a miles per hour. In how long a time will the last overtake the first? Discuss the result when $a > b$; $a = b$; $a < b$.

8. Two travellers set out from A, taking the same route and in the same direction. The first walks 9 hours per day, travels $5\frac{1}{2}$ miles per hour, and starts 2 days before the other. The second walks 12 hours per day, and travels $4\frac{1}{2}$ miles per hour. At what distance from A will the first be overtaken?

9. An express train which travels 42 miles per hour starts 50 minutes after a freight train, which it overtakes in 2 hours 5 minutes. What is the velocity of the freight train?

10. A messenger starts to carry a despatch; 5 days after his departure a second messenger is sent, who ought to overtake the first in 8 days, and to do this he is obliged to travel $2\frac{1}{2}$ more miles per day. How many miles per day did the first travel?

11. Two trains move in a line in the same direction. The first starts from A at 2 o'clock in the morning, and travels 8 miles per hour. The second starts from B at 9.30 o'clock in the morning, and travels $10\frac{1}{2}$ miles per hour. The distance between A and B is 20 miles. When and where do the two trains come together?

12. Two friends, at a distance of 78 miles, agree to meet in an intermediate locality, and set out at the same moment; one from A, travelling $5\frac{1}{4}$ miles per hour, the other from B, travelling $7\frac{1}{4}$ miles per hour. When and where do they meet?

13. At 12 o'clock the hands of a watch are together. At what time will they be opposite to each other?

14. How many times and at what hours will the two hands of a watch be together during the time between midnight and noon? The coincidence of the hands at midnight or at noon is not taken into account.

15. A hare is 50 leaps in advance of a fox; 3 leaps of the hare are equal to 2 leaps of the fox, but the fox makes only 3 leaps while the hare is making 4. How many leaps must the fox make to overtake the hare?

16. A company of foot-soldiers are 565 steps in advance of a squadron of cavalry. Allowing that a soldier takes 5 steps while a horse takes 4, but that 3 steps of a horse are equal to 4 of a man, how many steps must the horses of the squadron take to overtake the foot-soldiers?

17. If A, who is travelling, makes $\frac{1}{2}$ of a mile more per hour, he will employ only $\frac{4}{5}$ of the time; but if he makes $\frac{1}{2}$ of a mile less per hour, he will be on the route $2\frac{1}{2}$ hours more. Find the length of the route and the speed.

18. A boatman moves 5 miles in $\frac{3}{4}$ of an hour, rowing with the tide, and in returning it takes him $1\frac{1}{2}$ hours, rowing against a tide one-half as strong. What is the velocity of the stronger tide?

19. A boatman, rowing with the tide, moves n miles in t hours. Returning, he uses t' hours to accomplish the same distance, rowing against a tide m times as strong as the first. What is the velocity of the second tide?

20. A and B can accomplish a piece of work in m days; A and C can accomplish it in n days, and B and C in p days. How many days will it take each to do the work alone, and how many if they work together?

21. The circumference of a fore wheel of a carriage is a feet; that of a hind wheel, b feet. What distance will the carriage have passed over, when a fore wheel has made n more revolutions than a hind wheel?

22. Find a number such that if its pth part is added to it, the sum obtained will be equal to the qth part of the number, plus r.

23. A traveller wishes to accomplish a journey of k miles in n days; at the end of d days he receives news which obliges him to hasten by a days his return. How many more miles must he travel each day? Find the result if $k=360$; $n=18$; $d=7$; $a=5$.

24. A sum s is divided among A, B, C, in the following manner: the part given to A is to that given to B as $a:b$; that of B to that of C as $d:h$. What is the part given to each?

25. A number is divided by d, and the sum of the quotient, dividend, and divisor is b. What is this number? Find the result if $d=12$; $b=129$.

26. A wine merchant has two kinds of wine which he sells, one at a dollars, and the other at b dollars per gallon. He wishes to make a mixture of l gallons, which shall cost him on the average m dollars a gallon. How many gallons must he take of each?

Discuss the question (i.) when $a=b$; (ii.) when a or $b=m$; (iii.) when $a=b=m$; (iv.) when $a>b$ and $<m$; (v.) when $a>b$ and $b>m$.

27. A square pond, whose centre is distant from each side a meters, is surrounded by a fence. If the owner wishes to change the pond to a circular piece of water, what should be the radius that the fence may be just as long?

28. The population of a city increases each year t per cent of that of the preceding year; to-day it has A inhabitants. What was its population n years ago?

29. Three friends, A, B, and C, wish to divide a profit g that they have made in an enterprise. A has furnished a dollars during m months; B, b dollars during n months; and C, c dollars during p months. What will be the share of each?

30. Divide the number a into four parts such that, if n is added to the first, n subtracted from the second, the third multiplied by n, and the fourth divided by n, results will be equal. What are the results if $a=90$, and $n=2$.

31. If A can do a piece of work in $2m$ days, B and A together in n days, and A and C in $m+\frac{n}{2}$ days, how much time will it take them to do the work together?

32. If I divide a number by a, subtract b from the quotient, and multiply the remainder by m, I obtain the same result as if, from the number in question, I subtract n times its pth part. What is this number?

33. A man, setting out on a journey, drives at the rate of a miles per hour to the nearest railway station, distant b miles from his house. On arriving at the station, he finds that the express train for his place of destination has left c hours before. At what rate should he have driven, in order to reach the station just in time for the express train?

What does the answer become in the following cases: (i.) when $c=0$; (ii.) when $c=\frac{b}{a}$; (iii.) when $c=-\frac{b}{a}$. In case (ii.) how much time does the man have to drive from his house to the station? In case (iii.) what is the meaning of the *negative* value of c?

Ex. 20.

SIMPLE NUMERICAL EQUATIONS WITH TWO UNKNOWN QUANTITIES.

Solve:

1. $72x+14y=330$,
   $63x+7y=273$.

2. $2x+7y=52$,
   $3x-5y=16$.

3. $7x-4y=40$,
   $15x-13y=68$.

4. $4x+9y=79$,
   $7x-17y=40$.

5. $29x-14y=175$,
   $87x-56y=497$.

6. $171x-213y=642$,
   $114x-326y=244$.

7. $5x+9y=188$,
   $13x-2y=57$.

8. $5y-2x=21$,
   $13x-4y=120$.

9. $15x-7y=9$,
   $9y-7x=13$.

10. $12x+7y=176$,
    $3y-19x=3$.

11. $2x+5y=91$,
    $7y-3x=139$.

12. $4x+9y=106$,
    $8x+17y=198$.

13. $2x-7y=8$,
    $4y-9x=19$.

14. $17x+12y=59$,
    $19x-4y=153$.

15. $8x+3y=3$,
    $12x+9y=3$.

16. $69y-17x=103$,
    $14x-13y=-41$.

17. $2x+7y-34=0$,
    $5x+9y-51=0$.

18. $12x+13y-37=0$,
    $17x-19y-15=0$.

19. $3y-4x-1=0$,
    $18-3x=4y$.

20. $x=16-4y$,
    $y=34-4x$.

21. $2x=11+9y$,
    $3x-15=12y$.

22. $3y=104+x$,
    $5x=2y+78$.

23. $5x-3y=12,$
$19x-5y=73\frac{1}{3}.$

24. $5+3x=16y,$
$5x+28y=19.$

25. $21x+8y=-66,$
$28x-23y=13.$

26. $\frac{2x}{3}+\frac{3y}{2}=16\frac{1}{6},$
$\frac{3x}{2}-\frac{2y}{3}=16\frac{1}{6}.$

27. $\frac{7x}{4}+\frac{5y}{8}=20,$
$\frac{3x}{5}-\frac{y}{4}=5.$

28. $4+y=\frac{3x}{4},$
$x-8=\frac{4y}{5}.$

29. $\frac{2x}{3}=10-\frac{1}{2}y,$
$4\frac{3}{4}y=5x-7.$

30. $2\frac{4}{5}y-\frac{4}{5}x=90,$
$2\frac{1}{2}x+\frac{2}{5}y=90.$

31. $\frac{x}{4}+\frac{y}{6}=3\frac{1}{2},$
$\frac{x}{3}-\frac{y}{8}=\frac{1}{2}$

32. $\frac{x}{3}+\frac{y}{2}=\frac{4}{3},$
$\frac{x}{2}+\frac{y}{3}=\frac{7}{6}.$

33. $\frac{x}{4}+\frac{y}{4}=14,$
$\frac{x}{3}-\frac{y}{6}=1\frac{1}{6}.$

34. $\frac{x+y}{3}+\frac{y-x}{2}=9,$
$\frac{x}{2}+\frac{x+y}{9}=5.$

35. $\frac{x+y}{3}+x=15,$
$\frac{x-y}{5}+y=6.$

36. $\frac{x+y}{8}+\frac{x-y}{6}=5,$
$\frac{x+y}{4}-\frac{x-y}{3}=10.$

37. $\frac{x-1}{8}+\frac{y-2}{5}=2,$
$2x+\frac{2y-5}{3}=21.$

38. $\frac{3x-5y}{2}+3=\frac{2x+y}{5}.$
$8-\frac{x-2y}{4}=\frac{x}{2}+\frac{y}{3}.$

39. $\frac{4x+5y}{40}=x-y,$
$\frac{2x-y}{3}+2y=\frac{1}{2}.$

40. $\frac{x-2}{5}-\frac{10-x}{3}=\frac{y-10}{4},$
$\frac{2y+4}{3}=\frac{4x+y+13}{8}.$

41. $2+\frac{5x-6y}{13}=4y-3x,$
$12+\frac{5x-6y}{6}=2y+\frac{3x-2y}{4}.$

42. $\frac{5x-3}{4}-\frac{3x-19}{4}=2-\frac{3y-x}{6},$
$\frac{2x+y}{2}-\frac{9x-7}{8}=\frac{3y+9}{4}-\frac{4x+5y}{16}.$

43. $\frac{7+x}{5}-\frac{2x-y}{4}=3y-5,$
$\frac{5y-7}{2}+\frac{4x-3}{6}=18-5x.$

44. $\frac{1}{x}:\frac{1}{y}=13:11,$
$\frac{5x+3}{2}=\frac{7y-4}{3}.$

45. $\frac{3}{x+1}:\frac{4}{y+4}=1:\frac{2}{3},$
$\frac{3}{x}=\frac{4}{y}.$

46. $\frac{1}{x}+\frac{1}{y}=\frac{11}{30},$
$\frac{1}{x}-\frac{1}{y}=\frac{1}{30}.$

47. $\frac{1}{x}+\frac{1}{y}=6,$
$\frac{2}{x}-\frac{1}{y}=0.$

48. $\frac{3}{x}+\frac{8}{y}=3,$
$\frac{15}{x}-\frac{4}{y}=4.$

49. $\frac{1}{x}+\frac{2}{y}=10,$
$\frac{4}{x}+\frac{3}{y}=20.$

50. $\frac{7}{x}+\frac{5}{y}=19,$
$\frac{8}{x}-\frac{3}{y}=7.$

51. $\frac{5}{3x}+\frac{2}{5y}=7,$
$\frac{7}{6x}-\frac{1}{10y}=3.$

52. $\frac{4}{5x}+\frac{5}{6y}=5\frac{11}{15},$
$\frac{5}{4x}-\frac{4}{5y}=\frac{11}{20}.$

53. $\frac{1}{2x}+\frac{2}{3y}=3,$
$\frac{3}{4x}+\frac{4}{5y}=3\frac{9}{10}.$

54. $\frac{5x}{0.7}+\frac{0.15}{y}=6-\frac{0.15}{y},$
$\frac{10x}{3.5}+\frac{18}{y}=62.$

55. $\frac{x+y}{y-x}=5,$
$\frac{15x-4y}{6}-3=\frac{4x-3y}{5}+\frac{1}{5}.$

56. $x-\frac{2y-x}{23-x}=20-\frac{59-2x}{2},$
$y+\frac{y-3}{x-18}=30-\frac{73-3y}{3}.$

57. $\frac{6y+5}{8}-\frac{4x-5y+3}{4x-2y}=\frac{9y-4}{12},$
$\frac{8x+3}{4}-\frac{x-9y}{3-x}=\frac{6x-1}{3}.$

58. $\frac{7+8x}{10}-\frac{3(x-2y)}{2(x-4)}=\frac{11+4x}{5},$
$\frac{3(2y+3)}{4}=\frac{13}{4}+\frac{3y+4}{2}-\frac{3y+5x}{2(2y-3)}.$

59. $\frac{4x+7}{3}+\frac{5x-4y}{2x+1}=\frac{17+8x}{6},$
$\frac{5x-12}{4}-\frac{4x-6y-13}{2x-3y}=\frac{10x-53}{8}.$

60. $\frac{31-11y-7x}{12}+\frac{5x-3y+4}{3x-4y+5}=\frac{5y-4x}{3}-\frac{31y-9x-79}{12}.$
$21x+11y=202.$

61. $\frac{2(5-11x)}{11(x-1)}+\frac{11-7y}{3-y}=5,$
$\frac{7+2x}{3-x}-\frac{125-144y}{36(y+5)}=2.$

62. $\dfrac{7-6x}{10y-19}=\dfrac{4-3x}{5y-11}$,

$\dfrac{6x-10y-17}{3x-5y+2}=\dfrac{4x-14y-5}{2x-7y+12}$.

63. $\dfrac{12(7+6y)}{1-y}+\dfrac{432x-125}{3x+5}=72$,

$\dfrac{6(5-33y)}{3y-1}+\dfrac{11(11-21x)}{1-x}=165$.

64. $8y-\dfrac{4(4+15y)}{3x-1}=\dfrac{16xy-107}{2x+5}$,

$2+6x+9y=\dfrac{27y^2-12x^2+38}{3y-2x+1}$.

65. $x-\dfrac{2xy}{2y+5}=\dfrac{15x+4y}{6y-2x}+\dfrac{5x^2+4y^2+105}{(x-3y)(2y+5)}$,

$3-\dfrac{7x+2y}{5x}=5-\dfrac{5y+9}{3x}$.

66. $\dfrac{6+x}{6-y}=\dfrac{9}{4}$,

$\dfrac{4(x-2y)+5}{2}-2=\dfrac{2(5x^2-6y^2-7xy+x)}{5x+3y+3}$.

67. $\dfrac{1}{1-x+y}-\dfrac{1}{x+y-1}=\dfrac{2}{3}$,

$\dfrac{1}{\dfrac{1}{1-x+y}-\dfrac{1}{1-x-y}}=\dfrac{3}{4}$.

68. $\dfrac{1}{x+\dfrac{1}{y-\dfrac{5}{x}}}=\dfrac{1}{x-\dfrac{1}{y-\dfrac{7}{x}}}$,

$\dfrac{1}{y}\left(1-\dfrac{1}{x}\right)=1$.

Ex. 21.

LITERAL EQUATIONS WITH TWO UNKNOWN QUANTITIES.

Solve:

1. $x+y=a,$
   $x-y=b.$

2. $x+y=3a-2b,$
   $x-y=2a-3b.$

3. $2x-3y=5b-a,$
   $3x-2y=a+5b.$

4. $3x-2y=(a+b)^2+3ab,$
   $3y-2x=(a-b)^2-3ab.$

5. $ax+by=c,$
   $px+qy=r.$

6. $ax-by=m,$
   $cx+ey=n.$

7. $mx-ny=r,$
   $m'x+n'y=r'.$

8. $abx+cdy=2,$
   $ax-cy=\frac{d-b}{bd}.$

9. $a(3a+x)=b(b+y),$
   $ax+2by=d.$

10. $x-y=ab,$
    $cx+afy=ag.$

11. $\frac{x}{a}+\frac{y}{b}=c,$
    $\frac{x}{d}=y.$

12. $\frac{x}{a}-\frac{y}{b}=c,$
    $\frac{x}{g}-\frac{y}{f}=0.$

13. $\frac{x}{m}+\frac{y}{n}=p,$
    $\frac{x}{r}+\frac{y}{s}=q.$

14. $\frac{mx}{n}+\frac{py}{q}=t,$
    $\frac{x}{n}-\frac{ry}{s}=v.$

15. $\frac{ax}{b}+\frac{cy}{d}=\frac{f}{g},$
    $\frac{mx}{p}+\frac{ny}{q}=\frac{s}{t}.$

16. $\frac{x}{a}+\frac{y}{b}=1-\frac{x}{c},$
    $\frac{y}{a}+\frac{x}{b}=1+\frac{y}{c}.$

17. $\frac{x}{m+n}-\frac{y}{m-n}=\frac{1}{m+n},$
    $\frac{x}{m+n}+\frac{y}{m-n}=\frac{1}{m-n}.$

18. $\dfrac{x}{m-a}+\dfrac{y}{m-b}=1,$
$\dfrac{x}{n-a}+\dfrac{y}{n-b}=1.$

19. $ax+by=2a^2-2b^2,$
$\dfrac{y}{a-b}-\dfrac{x}{a+b}=\dfrac{a^2+b^2}{ab}.$

20. $(m+n)x-(m-n)y=4mn,$
$\dfrac{x}{m+n}+\dfrac{y}{m-n}=2.$

21. $\dfrac{x}{q^2-1}-\dfrac{y}{p^2-1}=p^2-q^2,$
$\dfrac{x}{p^2+1}+\dfrac{y}{q^2+1}=p^2+q^2-2.$

22. $\dfrac{x+y-1}{x-y+1}=m,$
$\dfrac{y-x+1}{x-y+1}=mn.$

23. $\dfrac{x-m}{y}=\dfrac{p}{q},$
$\dfrac{m}{x}=\dfrac{n}{y}.$

24. $\dfrac{x+m}{y-n}=\dfrac{p}{q},$
$qx+py=s.$

25. $\dfrac{a}{b+y}=\dfrac{b}{a-x},$
$\dfrac{c}{d-x}=\dfrac{d}{c+y}.$

26. $\dfrac{1}{x}+\dfrac{1}{y}=m,$
$\dfrac{1}{x}-\dfrac{1}{y}=n.$

27. $\dfrac{m}{x}+\dfrac{1}{y}=p,$
$\dfrac{n}{x}+\dfrac{1}{y}=q.$

28. $\dfrac{m}{x}+\dfrac{n}{y}=p,$
$\dfrac{r}{x}+\dfrac{s}{y}=t.$

29. $\dfrac{x}{m}+\dfrac{y}{n}=p,$
$\dfrac{r}{x}=\dfrac{q}{y}.$

30. $\dfrac{dy}{bx}=\dfrac{c}{a},$
$bx+dy=a+c.$

31. $\dfrac{x+y}{x-y}=\dfrac{a}{b-c},$
$\dfrac{x+c}{y+b}=\dfrac{a+b}{a+c}.$

32. $\dfrac{x-m}{y-m}=\dfrac{m-n}{m+n}$,
$\dfrac{x}{y}=\dfrac{m^3-n^3}{m^3+n^3}$.

34. $\dfrac{x+y+1}{x-y+1}=\dfrac{p+1}{p-1}$,
$\dfrac{x+y+1}{x-y-1}=\dfrac{1+m}{1-m}$.

33. $\dfrac{x+1}{y+1}=\dfrac{m+n+p}{m-n+p}$,
$\dfrac{x-1}{y-1}=\dfrac{m+n-p}{m-n-p}$.

35. $\dfrac{b}{x-a+c}=\dfrac{c}{y-a+b}$,
$\dfrac{x+c}{a+b}=\dfrac{y+b}{a+c}$.

36. $\dfrac{bx+ay}{a+b+2}-x=\dfrac{b-a-1}{b}$,
$bx-a=ay-b$.

37. $\dfrac{1+x}{2}=\dfrac{y-1}{2}+\dfrac{(a-b)^2-2b^2}{a^2-b^2}$,
$by-ax=\dfrac{(3a+b)ab}{a^2-b^2}+\dfrac{ab}{a+b}-(a+b)$.

38. $3x+5y=\dfrac{2(4b-m)bm}{b^2-m^2}$,
$b^2x-\dfrac{bcm^2}{b+m}+(b+c+m)my=m^2x+(b+2m)bm$.

39. $(p-q)x=2p-(p+q)y$,
$2x-\dfrac{5p^2-2pq+5q^2}{p^2-q^2}=-3y$.

40. $2x+3y=\dfrac{2m^2+mn+3n^2}{m^2-n^2}$,
$3x-\dfrac{3m^2-5mn-2n^2}{m^2-n^2}=2y$.

41. $0.6x+y=\dfrac{a}{b}+\dfrac{b(3b-2a)}{5ab}$,
$x-0.6y=\dfrac{5b^2+a(8b-3a)}{5ab}$

42. $1.5x+y=\frac{m^2+n^2}{2mn}$,

$2x-3y=\frac{(2m+3n)(2m-3n)}{6mn}+2.166.....$

43. $x=\frac{q(q-ry)+(2pq-r)r}{qr}$,

$rx+qy=q(p+1)+r(p-1)$.

44. $qx-py=2m$,

$3y=\frac{pq[5(p-x)+3q]+(5p-3q)m}{pq}$

45. $\frac{(c+d)x}{5}-\frac{(c-d)y}{3}=4cd$,

$3x+\frac{5(c+d)y}{c-d}=30(c+d)$.

46. $x+y=\frac{2bc(a^3-2a^2b+3a^2c)}{abc-2b^2c+3bc^2}$,

$a(x-a^2)+b(y+b^2)=ab(a+b)+(a-b)^2$.

47. $x+y=\frac{4(a^2-2ab+ac+b^2-bc)}{a-b}$,

$3a(x-a)=3ac-6ab+4bc+2c(y-c)$.

48. $x+y=\frac{2[2m(m+p)-n(m+n+2p)]}{m-n}$,

$m(x-m)=5mp+3mn+2p^2-p(n+y)$.

49. $\frac{2(m^2-n^2)}{x}-m=\frac{ny}{x}$,

$\frac{1}{(m-n)x}-\frac{1}{(m+n)y}=\frac{m^2+n^2}{mnxy}$.

50. $2+x-y=\dfrac{2[(c-d)^2-2d^2]}{c^2-d^2}$,

$dy-\dfrac{cd(3c+d)}{c^2-d^2}=cx-(c+d)+\dfrac{cd}{c+d}$.

51. $x^2-y^2=m$,
$x-y=n$.

52. $x^2-y^2=m^2$,
$x-y=n$.

53. $\dfrac{x}{y}=\dfrac{c+d-\dfrac{cd}{c+d}}{c-d+\dfrac{cd}{c-d}}$,
$x+y=2c^3$.

54. $\dfrac{x}{y}=\dfrac{a-b+\dfrac{b^2}{a-b}\left(1-\dfrac{b(a+b)}{a^2+ab+b^2}\right)}{\dfrac{a^2}{a+b}+\dfrac{b^3}{a^2+ab+b^2}}$,
$x-y=2b^5$.

Ex. 22.

Numerical Equations with More than Two Unknown Quantities.

Solve:

1. $x+y=5$,
$y+z=7$,
$x+z=6$.

2. $2x+y=5$,
$x+3z=16$,
$5y-z=10$.

3. $2x+3y=21$,
$3x-2z=-5$,
$4y+7z=69$.

4. $5x-3y=-1$,
$9y-2z=12$,
$8x+3z=17$.

5. $0.5x+0.3y=2.2$,
$1.2z-8.5y=-24.4$,
$3.3z+1.3x=29$.

6. $0.25x+0.125y=5$,
$0.3y-0.9z=-21$,
$1.4x+1.2z=50$.

7. $2.5x+1.5y=6.5$,
$3.5x-1.5z=4$,
$0.6y+z=2.2$.

8. $0.65x-0.95y=0.5$,
$5.1y-3.3z=6$,
$20.3z-14.7x=21$.

9. $\frac{x}{2}+\frac{y}{3}=12,$
$\frac{z}{5}-\frac{y}{6}=4,$
$\frac{x}{12}+\frac{z}{7}=6.$

10. $\frac{2}{5}x-y=0,$
$x-\frac{3}{2}z=3,$
$\frac{1}{2}z-\frac{1}{3}y=2.$

11. $\frac{x}{2}+\frac{3y}{4}=3,$
$\frac{3x}{2}+z=5\frac{1}{2},$
$\frac{3y}{8}+\frac{z}{2}=1\frac{1}{4}.$

12. $\frac{2x}{3}+\frac{3y}{4}=10,$
$\frac{5y}{24}-\frac{z}{5}=-\frac{1}{3},$
$\frac{5x}{6}+\frac{7z}{10}=12.$

13. $x+y+z=36,$
$3x+6=3z,$
$6y-7=5z.$

14. $x-y+z=34,$
$2y=5x+8,$
$3z-12=13x.$

15. $x+y+z=100,$
$0.7x-4=y,$
$0.3x+4=z.$

16. $\frac{5x}{2}+\frac{10y}{3}+\frac{17z}{4}=64,$
$3\frac{3}{4}x=2\frac{1}{2}y,$
$3\frac{1}{3}y=2\frac{1}{2}z.$

17. $x+6=2\frac{1}{3}y,$
$y+1=3.5z,$
$z+8=1.25x.$

18. $x=21-4y,$
$z=9-\frac{2}{3}x,$
$y=64-7.5z.$

19. $1.7x-0.6y=-4.08,$
$2.8x+0.6z=-6.72,$
$2.8y-1.7z=19.04.$

20. $x+y+z=35,$
$y-x+z=-5,$
$z-y-x=-25.$

21. $x+y-z=17,$
$x+z-y=13,$
$y+z-x=7.$

22. $x+y+z=6,$
$5x+4y+3z=22,$
$15x+10y+6z=53.$

23. $x+y+z=30,$
$8x+4y+2z=50,$
$27x+9y+3z=64.$

24. $x+y+z=24,$
$x:y=7:8,$
$\frac{y}{z}=\frac{8}{9}.$

25. $2x+3y-z=130$,
$x:y=1:5$,
$\frac{y}{z}=\frac{5}{7}$.

26. $7x-5y+3z=66$,
$x:y=2:3$,
$\frac{y}{z}=\frac{3}{4}$.

27. $x+y+z=12$,
$x+z=3+5$,
$y+z=12+15$.

28. $5x+7y-2z=13$,
$8x+3y+z=17$,
$x-4y+10z=23$.

29. $4x-5y+2z=6$,
$2x+3y-z=20$,
$7x-4y+3z=35$.

30. $8x+4y-3z=6$,
$x+3y-z=7$,
$4x-5y+4z=8$.

31. $12x+5y-4z=29$,
$13x-2y+5z=58$,
$17x-y-z=15$.

32. $4x-3y+2z=40$,
$5x+9y-7z=47$,
$9x+8y-3z=97$.

33. $3x+2y+z=23$,
$5x+2y+4z=46$,
$10x+5y+4z=75$.

34. $4x-2y+5z=18$,
$2x+4y-3z=22$,
$6x+7y-z=63$.

35. $2x+5y-7z=-288$,
$5x-y+3z=227$,
$7x+6y+z=297$.

36. $x+y+z=6$,
$8x+4y+2z=22$,
$27x+9y+3z=54$.

37. $0.1x+0.2y+0.3z=14$,
$0.4x+0.5y+0.6z=32$,
$0.7x-0.8y+0.9z=18$.

38. $1.5x-2.5y+2z=2.5$,
$3.5x+y-1.5z=1$,
$2x+1.5y-0.5z=3.5$.

39. $\frac{x}{2}+\frac{y}{3}+\frac{z}{4}=62$,
$\frac{x}{3}+\frac{y}{4}+\frac{z}{5}=47$,
$\frac{x}{4}+\frac{y}{5}+\frac{z}{6}=38$.

40. $258-\frac{x}{5}=\frac{y}{7}+\frac{z}{9}$,
$304-\frac{z}{5}=\frac{x}{7}+\frac{y}{9}$,
$296-\frac{z}{7}=\frac{y}{5}+\frac{x}{9}$.

41. $3x-\frac{1}{4}y+z=7.5$,
$2x-\frac{y-3z}{3}=5\frac{1}{3}$,
$2x-\frac{1}{2}y+4z=11$.

42. $\frac{1}{x+y}=1,$
$\frac{2}{x+z}=1,$
$\frac{3}{y+z}=1.$

43. $\frac{2(5y-x)}{3}=11\frac{1}{3},$
$\frac{x-3y}{2x+z}+4=\frac{35}{11},$
$\frac{7y+8z}{2}=34.$

44. $\frac{2(3y-2x)}{3z-7}=1,$
$\frac{5z-x}{2y-3z}=1,$
$\frac{y-2z}{3y-2x}=1.$

45. $x+2y:3y+4z=7:8,$
$3y+4z:5x+6z=8:9,$
$x+y-z=126.$

46. $3y+7z:5x-4y=107:5,$
$5x-4y:7x-2z=5:41,$
$3x-5y+4z=21.$

47. $\frac{3x+5y-7}{5y-z+8}=\frac{14}{19},$
$\frac{3x+5y-7}{7z+3y-21}=\frac{7}{8},$
$2x+3y+4z=29.$

48. $\frac{5-4x}{6y+1}=\frac{55-2x}{3y+74},$
$\frac{3+x}{3z-3}=\frac{2x+9}{6z},$
$\frac{3y-1}{3z+1}=\frac{7y-3}{7z-1}.$

49.* $\frac{12}{2x+3y}-\frac{7.5}{3x+4z}=1,$
$\frac{30}{3x+4z}+\frac{37}{5y+9z}=3,$
$\frac{222}{5y+9z}-\frac{8}{2x+3y}=5.$

50.† $115(113-x)+719(y-219)-590(337-z)=27,$
$\frac{5(113-x)+2}{2(y-219)}=2,$ $\frac{337-z}{y-221}=4.$

51. $\frac{7x-3y+6}{20}-\frac{7y+2z-10}{45}=\frac{5z-11x-32}{36}+2,$
$71z-10x+235+27y=345x-189y+185,$
$5y+3z=44.$

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* Put $\frac{1}{2x+3y}=m;$ $\frac{1}{3x+4z}=n;$ $\frac{1}{5y+9z}=p.$

† Put $113-x=m;$ $y-219=n;$ $337-z=p.$

52. $\frac{5x-7y+2}{12}-\frac{8x+3z-4}{21}=\frac{11y-5z-4x+18}{14}$,

$\frac{11x-5z+12}{14}-\frac{3y+7z-2x}{18}=\frac{8z-3x+82}{21}$,

$3x-y-2z=16$.

53. $\frac{6x-7y+8z-7}{25}-\frac{8x-3z+44}{35}$

$=\frac{13y-7z+1}{15}+\frac{4x-5y+23}{21}$,

$\frac{4x+5z-1}{33}-\frac{3y-5x+60}{22}=\frac{9z-7y+2}{4}+\frac{7x-6y-5z-8}{6}$,

$2x+y-z=2$.

54. $\frac{3x-7y+5}{18}+\frac{7-4y-11z}{9}=\frac{5x-7z}{8}-\frac{18x+27y+11z-30\frac{1}{2}}{36}$,

$\frac{4x-9y+11}{22}-\frac{5y-3x-9}{4}=\frac{6y+5z}{11}+\frac{7x-17y-3z+19\frac{6}{11}}{8}$,

$\frac{5y+6z-17}{54}-\frac{2y+7z+5}{9}=\frac{x+9y}{27}-\frac{2x+7y+11z+18\frac{1}{2}}{18}$.

55. $\frac{1}{x}+\frac{1}{y}=1$,

$\frac{1}{x}+\frac{1}{z}=2$,

$\frac{1}{y}+\frac{1}{z}=\frac{3}{2}$.

56. $\frac{2}{x}+\frac{1}{y}=\frac{3}{z}$,

$\frac{3}{z}-\frac{2}{y}=2$,

$\frac{1}{x}+\frac{1}{z}=\frac{4}{3}$.

57. $\frac{1}{x}+\frac{2}{y}-\frac{3}{z}=1$,

$\frac{5}{x}+\frac{4}{y}+\frac{6}{z}=24$,

$\frac{7}{x}-\frac{8}{y}+\frac{9}{z}=14$.

58. $\frac{3}{x}-\frac{4}{5y}+\frac{1}{z}=7\frac{3}{5}$,

$\frac{1}{3x}+\frac{1}{2y}+\frac{2}{z}=10\frac{1}{6}$,

$\frac{4}{5x}-\frac{1}{2y}+\frac{4}{z}=16\frac{1}{10}$.

59.* $\dfrac{xy}{x+y}=\dfrac{1}{5}$,
$\dfrac{xz}{x+z}=\dfrac{1}{6}$,
$\dfrac{yz}{y+z}=\dfrac{1}{7}$.

60.† $\dfrac{xy}{4y-3x}=20$,
$\dfrac{xz}{2x-3z}=15$,
$\dfrac{yz}{4y-5z}=12$.

61. $x+2y=9$,
$3y+4z=14$,
$7z+u=5$,
$2u+5x=8$.

62. $2x-9y+3z-10u=121$,
$x+7y-z-u=583$,
$3x+2y+5z+2u=255$,
$4x-6y-2z-9u=516$.

63. $x+2y+3z+4u=2$,
$2x+3y+4z+5u=3$,
$6x+7y-8z+9u=3$,
$256x+64y+16z+4u=100$.

64. $3u-2y=2$,
$2x+3y=39$,
$5x-7z=11$,
$4y+3z=41$.

65. $2x-3y+2z=13$,
$4u-2x=30$,
$4y+2z=14$,
$5y+3u=32$.

66. $\dfrac{x}{3}+\dfrac{y}{5}+\dfrac{2z}{7}=58$,
$\dfrac{5x}{4}+\dfrac{y}{6}+\dfrac{z}{3}=76$,
$\dfrac{x}{2}+\dfrac{3z}{8}+\dfrac{u}{5}=79$,
$\dfrac{y}{5}+\dfrac{z}{2}+\dfrac{u}{25}=92$.

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* By taking the reciprocal values, and performing the division, the equation will become $\dfrac{x+y}{xy}=\dfrac{5}{1}$, or $\dfrac{1}{y}+\dfrac{1}{x}=5$, etc.

† By taking the reciprocal values, and performing the division, the equation will become $\dfrac{4}{x}-\dfrac{3}{y}=\dfrac{1}{20}$, etc.

Ex. 23.

Literal Equations with More than Two Unknown Quantities.

Solve:

1. $x+y=a,$
$x+z=b,$
$y+z=c.$

2. $mx+y=n,$
$z-y=p,$
$qx+z=t.$

3. $x+y+z=m,$
$x-y+z=n,$
$x+y-z=p.$

4. $x+ay+a^2z+a^3=0,$
$x+by+b^2z+b^3=0,$
$x+cy+c^2z+c^3=0.$

5. $ax+by-cz=2ab,$
$by+cz-ax=2bc,$
$cz+ax-by=2ac.$

6. $x-mx+y=0,$
$y-ny+z=0,$
$x+z=p.$

7. $x+y+z=a,$
$\frac{x}{y}=\frac{m}{n},$
$\frac{y}{z}=\frac{p}{q}.$

8. $\frac{1}{x}=m-\frac{1}{y},$
$\frac{1}{x}=n-\frac{1}{z},$
$\frac{1}{y}=p-\frac{1}{z}.$

9. $\frac{a}{x}+\frac{b}{y}-\frac{c}{z}=m,$
$\frac{a}{x}-\frac{b}{y}+\frac{c}{z}=n,$
$-\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=p.$

10. $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=3,$
$\frac{a}{x}+\frac{b}{y}-\frac{c}{z}=1,$
$\frac{2a}{x}-\frac{b}{y}-\frac{c}{z}=0.$

11. $\frac{1}{x+y}=m,$
$\frac{1}{x+z}=n,$
$\frac{1}{y+z}=p.$

12. $\dfrac{xy}{x+y}=a,$ $\dfrac{xz}{x+z}=b,$ $\dfrac{yz}{y+z}=c.$

13. $\dfrac{x}{a}+\dfrac{y}{b}+\dfrac{z}{c}=1,$ $\dfrac{x}{a}+\dfrac{y}{c}+\dfrac{z}{b}=1,$ $\dfrac{x}{b}+\dfrac{y}{a}+\dfrac{z}{c}=1.$

14. $\dfrac{x}{m+n}+\dfrac{y}{n-p}+\dfrac{z}{m+p}=2p,$
$\dfrac{x}{m-n}-\dfrac{y}{n-p}+\dfrac{z}{p-m}=2m,$
$\dfrac{x}{m-n}-\dfrac{y}{n-p}+\dfrac{z}{m+p}=2m-2p.$

15. $\dfrac{x}{m+n}+m=n-\dfrac{y}{n+p},$
$\dfrac{y}{p-m}-m=p-\dfrac{z}{m+p},$
$\dfrac{x}{n-p}+p=n+\dfrac{z}{m-n}.$

16. $m-\dfrac{x}{n+p}=\dfrac{y}{p-m}-n,$
$n-\dfrac{y}{p+m}=\dfrac{z}{m-n}-p,$
$p-\dfrac{z}{m-n}=\dfrac{x}{n-p}-m.$

17. $\dfrac{m-n}{nx+my}=\dfrac{(n-p)(m-p)}{p},$
$\dfrac{n-p}{py+nz}=\dfrac{(p-m)(n-m)}{m},$
$\dfrac{p-m}{mz+px}=\dfrac{(m-n)(p-n)}{n}.$

18. $2np = (p+m)x-(p-m)y,$
$2mp = (m+n)y-(m-n)z,$
$2mn = (n+p)z-(n-p)x.$

19. *$(z+x)m-(z-x)n = 2yz,$
$(x+y)n-(x-y)p = 2xz,$
$(y+z)p-(y-z)m = 2xy.$

20. $y+z+u = m,$
$z+u+x = n,$
$u+x+y = p,$
$x+y+z = q.$

21. $x+y-u = m,$
$y+z-v = n,$
$z+u-x = 0,$
$u+v-y = p,$
$v+x-z = q.$

22. $x+y+z = 3m+n+p,$
$x+y+u = m+3n+p,$
$x-y-u = m-n+p,$
$y+z-u = m-n-p.$

23. $x+y+z+u+v = l,$
$x+y+z+v+w = m,$
$x+y+z+u+w = n,$
$x+y+v+u+w = 0,$
$x+z+v+u+w = p,$
$y+z+v+u+w = q.$

Ex. 24.

PROBLEMS WITH TWO OR MORE UNKNOWN QUANTITIES.

1. A man has two vessels, and for the two a single cover worth \$0.90 If he puts the cover upon the first vessel it will be worth $1\frac{1}{2}$ times as much as the other. If he puts the cover upon the second vessel it will be worth $1\frac{1}{1\frac{1}{2}}$ times as much as the first. What is the value of each vessel?

* Do not determine the values of x, y, and z, but those of the known quantities, in terms of the unknown; then use the three new equations in order to deduce x, y, and z. In replacing the unknown quantities, x, y, and z, by the letters m, n, and p, and *vice versa* we can obtain the equations of Example 20.

2. Two numbers are such that 3 times the first plus 5 times the second equals 44; but 3 times the second plus 6 times the first equals 60. What are the numbers?

3. A book-seller sells 3 bound copies of a work and 7 stitched copies. He receives for them all $32.40. Another day he sells 2 bound copies and 5 stitched copies for $22.60. How much does a stitched copy cost, and how much a bound copy?

4. A man has two horses, and a saddle which is worth $150. If he puts the saddle upon the first horse, he is worth double the other; if he puts the saddle upon the second horse, he is still worth $300 less than the first. What is the value of each horse?

5. A fraction is such that if 1 is added to the numerator, its value is 1; and if, after having doubled the numerator, 1 is added to the denominator, it still equals 1, what is the fraction?

6. A fraction is such that if 3 is added to each of its terms it equals $\frac{4}{5}$; and if 3 is subtracted from each of its terms it equals $\frac{1}{2}$. What is the fraction?

7. The sum of two numbers is 210, and their sum is to the first as 7 : 4. Find the numbers.

8. Find two numbers whose sum is 54, and whose sum and difference are in the ratio of 9 : 5.

9. A number consists of two digits whose difference is 3. If the order of the digits is changed, the number obtained is $\frac{4}{7}$ of the first number. What is the first number?

10. A number consists of two digits whose sum is 8. If the order of the digits is changed, the new number is $\frac{5}{4}$ of the sum of the digits plus 2 times the first number. What is the first number?

11. The sum of two figures of which a number is composed is 9. If 3 is subtracted from each of the figures, the result is half of the first number diminished by 6. What is the first number?

12. A number of two figures is such that divided by their difference the quotient is 12; but if the order of the figures is changed, and 9 added to the number thus formed, and the result divided by the sum of the figures, 8 is the quotient; find the number.

13. The sum of the two figures of a number equals 6 times their difference, and the number itself is 3 more than 6 times the sum of the figures. Find the number.

14. The sum of two numbers is 26. The third of the first and three-fourths of the second are equal. What are the numbers?

15. Two numbers of three figures each are such that their sum, increased by 1, is 1000. If the two sets of figures are written one after the other, the number obtained by placing the figures of the largest number first is 6 times that obtained by placing the figures of the smallest number first. What are the numbers?

16. Find two numbers such that the first added to 4 times the second equals 29; the second added to 6 times the first equals 36.

17. Two numbers are such that the smaller divided by the greater gives for a quotient $\frac{5}{11}$, and the greater divided by the smaller gives 2 for a quotient and 5 for a remainder. Find the numbers.

18. The smaller of two numbers divided by the larger is 0.21 with a remainder 0.04162. The greater divided by the smaller is 4 with 0.742 for a remainder. What are these numbers?

19. Find two numbers whose sum and difference are as 5 : 1, and their sum and product are as 5 : 8.

20. A servant receives $9 to buy 4 kilograms of butter and 7 of soap, and 15 cents ought to be returned. But she makes a mistake and buys 7 kilograms of butter and 4 of soap, and requires 30 cents more. What was the price of the butter and the soap?

21. If $28 are paid to 6 artificers and 2 laborers for a day's work; and if for another day's work the same sum is paid to 5 artificers and 4 laborers; what is a day's pay of an artificer and of a laborer?

22. A pedestrian has a certain distance to walk. After having passed over 20 miles, he increases his speed by 1 mile per hour. If he had walked the entire journey with this speed, he would have accomplished his walk in 40 minutes less time; but, by keeping his first pace, he would have arrived 20 minutes later than he did. What distance had he to walk?

23. A garden is of a rectangular form. If 3 meters are added to its length, and 2 meters to its width, the surface will be increased by 66 square meters; but if 2 meters are added to its length, and 3 to its width, the original surface will be increased by 71 meters. What are the two dimensions of the garden?

24. If 5 grams of an ingot are mixed with 11 grams of a second ingot, the alloy formed is at the standard of $0.868\frac{3}{4}$. But if 2 grams of the first and 1 gram of the second are taken, the alloy will be at the standard of $0.833\frac{1}{3}$. What is the standard of the two ingots?

25. A cask contains 12 gallons of wine and 18 gallons of water; a second contains 9 gallons of wine and 3 of water. How many gallons must be taken from each cask, so that when mixed there may be 14 gallons, consisting half of water, and half of wine?

26. Two persons owe $1200 each. The first said to the second, "If you give me $\frac{3}{4}$ of what you have, I shall have enough to pay my debt." The second replied, "If you give me $\frac{3}{5}$ of what your purse contains, I can pay my debt." How much do they each have?

27. A society contains a certain number of men and women. When 4 women leave, there are $1\frac{1}{2}$ times as many men as women. But when the women return with their husbands, the number of men is $\frac{1}{8}$ greater than that of the women. How many men and women did the society have?

28. Two friends, A and B, in reviewing their libraries, said that $\frac{1}{5}$ of A's library plus $\frac{1}{6}$ of B's would be 780 volumes; but that $\frac{1}{6}$ of A's library added to $\frac{1}{5}$ of B's would be 20 volumes less. How many volumes did each have?

29. Two women buy velvet and silk. One buys $3\frac{1}{2}$ yards of velvet and $12\frac{3}{4}$ yards of silk; the other took $4\frac{1}{2}$ yards of velvet and 5 yards of silk. They each pay $63.80. How much per yard did each cost?

30. Three brothers, M, N, P, have bought a house which cost $32,000. M could pay the whole sum if N would give him $\frac{5}{8}$ of what he has; N could pay it if P would give him $\frac{3}{5}$ of what he has; and P could pay the whole sum if he had $\frac{1}{2}$ of what M has plus $\frac{3}{16}$ of what N had. How much does each have?

31. Three women go to market to sell eggs. If the first gives $\frac{1}{7}$ of hers, and the third $\frac{1}{18}$ of hers to the second, they will all have an equal number; they all together carry 360. How many has each?

32. A farmer sold at the same prices, first, 5 hektoliters of wheat, 4 of rye, and 3 of oats for \$24.20; second, 3 hektoliters of wheat, 5 of rye, and 4 of oats for \$21.40; third, 1 hektoliter of wheat, 8 of rye, and 5 of oats for \$22.20. What was the price per hektoliter of each kind of grain?

33. A number is composed of three figures whose sum is 17. The figure of the hundreds is double that of the units. When 396 is subtracted the order of the figures is reversed. What is the number?

34. Find three numbers such that the first plus half the sum of the other two is 120; that the second, added to $\frac{1}{5}$ of the difference between the third and the first, is 70; and half the sum of the three equals 95.

35. Find three numbers such that the first plus $\frac{1}{2}$ the second, the second plus $\frac{1}{3}$ of the third, and the third plus $\frac{1}{4}$ of the first are each 1000.

Ex. 25.

Square and Cube Roots.

Extract the square root to the fifth decimal place:

1. 326.	6. 1020.	11. 9.3434.....
2. 29.	7. 935.	12. 27.1919.....
3. 491.	8. 842.	13. 3.666.....
4. 726.	9. 719.	14. 3.1414.....
5. 391.	10. 145.	15. 1.315315.....

Extract the square root to the third decimal:

16. 120.
17. 133.
18. 3602.
19. 401.
20. 101.
21. 10,004.
22. 9997.
23. 90,004.
24. 1,440,009.
25. 809,995.
26. 6396.
27. 40,003.

Extract the square root of:

28. $4+9y^2-20x+25x^2+30xy-12y$.

29. $4x^4+9y^6-12x^2y^3+16x^2z^5+16z^{10}-24y^3z^5$.

30. $4+13a^2+9a^4-4a-6a^3$.

31. $a^2b^2+2a^2b+a^2-2ab^2-2ab+b^2$.

32. $18x^2+x^4+1-8x^3-8x$.

33. $9a^4+25b^2+64m^2+z^2-30a^2b+48a^2m-6a^2z$
$-80bm+10bz-16mz$.

34. $y^{10}+4u^6+9v^8+6v^4y^5-4u^3y^5+16q^{10}-8q^5y^5$
$-12u^3v^4+16q^5u^3-24q^5v^4$.

35. $a^4+16d^{4x}+9e^{2y}+9c^{10}-6a^2c^5-8a^2d^{2x}+24c^5d^{2x}$
$+6a^2e^y-18c^5e^y-24d^{2x}e^y$.

36. $9a^6-12a^5b-38a^4b^2+52a^3b^3+33a^2b^4-56ab^5+16b^6$.

37. $\frac{m^4}{4}+\frac{n^6}{9}+\frac{m^2p^4}{4}-\frac{m^2n^3}{3}+\frac{p^8}{16}-\frac{n^3p^4}{6}$.

38. $\frac{x^2}{y^2}-\frac{4xz}{uy}+\frac{4z^2}{u^2}+\frac{6qx}{vy}+\frac{9q^2}{v^2}-\frac{12qz}{uv}$.

39. $\frac{9}{25}+\frac{a^8}{25}+\frac{m^6}{36}-\frac{6a^4}{25}-\frac{m^3}{5}+\frac{4n^8}{49}+\frac{12n^4}{35}+\frac{a^4m^3}{15}-\frac{4a^4n^4}{35}$
$-\frac{2m^3n^4}{21}$.

40. $\frac{9m^6n^4}{25p^6q^8} - \frac{12m^5n^5}{35p^7q^9} - \frac{332m^4n^6}{735p^8q^{10}} + \frac{16m^3n^7}{63p^9q^{11}} + \frac{16m^2n^8}{81p^{10}q^{12}}$.

41. $25\frac{8}{7} - \frac{20x}{7y} + \frac{9y^2}{16x^2} - \frac{15y}{2x} + \frac{4x^2}{49y^2}$.

42. $x^8 - \frac{2x^{11}}{a^3} + 2a^4x^4 + \frac{x^{14}}{a^6} - 2ax^7 + a^8$.

43. $x^6y^6 - \frac{2x^3y^{12}}{3a^3} + \frac{y^{18}}{9a^6} + \frac{4x^3y^{18}}{a^9} - \frac{4y^{24}}{3a^{12}} + \frac{4y^{30}}{a^{18}}$.

Extract the cube root to the third decimal:

44. 867.	49. 4915.	54. 3.2525.....
45. 1241.	50. 8972.	55. 317.8181.....
46. 976.2	51. 12,761.	56. 15.4141.....
47. 517.	52. 56,431.	57. 690.
48. 1637.	53. 20,911.	58. 10,010.

Extract the cube root of

59. $8b^3 - 9a^2y + 6a^2b + a^3 + 27ay^2 + 12ab^2 - 27y^3 + 54y^2b - 36yb^2 - 36aby$.

60. $147x^2v - 126xuv + 343x^3 - 441x^2u - 27u^3 + v^3 + 189xu^2 + 21xv^2 + 27u^2v - 9uv^2$.

61. $125a^9 - 8b^6 + c^{12} - 150a^6b^2 + 75a^6c^4 + 60a^3b^4 + 15a^3c^8 + 12b^4c^4 - 6b^2c^8 - 60a^3b^2c^4$.

62. $36x^{10}y^3 - 24x^{10}z^3 + 54x^5y^4 + 8x^{15} + 27y^6 + 24x^5z^6 - 54y^4z^3 - 72x^5y^2z^3 - 8z^9 + 36y^2z^6$.

63. $162a^6d^7 - 108a^4b^5d^7 + 27a^{12} - b^{15} + 9a^4b^{10} + 324a^4d^{14} + 216d^{21} + 18b^{10}d^7 - 108b^5d^{14} - 27a^8b^5$.

64. $360m^3np + 1000m^6 - 900m^4n - 27n^3 - 600m^4p - 8p^3 + 270m^2n^2 + 120m^2p^2 - 54n^3p - 36np^2$.

65. $36a^5b^4 + 8a^6b^3 - 45a^3b^6 - 8b^9 + 30a^4b^5 - 30a^2b^7 + 36ab^8$.

66. $225x^7y^3 - 150x^5y^5 + 27x^3y^3 - 8x^6y^9 + 36x^5y^7 + 125x^9y^3 + 60x^7y^7 + 135x^5y^3 - 180x^6y^5 - 54x^4y^5$.

67. $125m^3n^6 - 192m^7n^{10} - 64m^9n^{12} + 240m^5n^8 - 504m^6n^9 + 450m^4n^7 + 288m^8n^{11}$.

68. $-172a^6x^9 + 27a^3x^6 + 54a^4x^7 - 125a^9x^{13} - 99a^5x^8 + 165a^7x^{10} + 150a^8x^{11}$.

69. $\frac{8a^6}{x^3} - \frac{36a^7}{x^4} + \frac{a^{12}}{x^9} + \frac{33a^{10}}{x^7} + \frac{66a^8}{x^5} - \frac{9a^{11}}{x^8} - \frac{63a^9}{x^6}$.

70. $\frac{3m^8}{y^{14}} - \frac{m^3}{y^9} + \frac{m^9}{y^{15}} - \frac{5m^6}{y^{12}} + \frac{3m^4}{y^{10}}$.

71. $\frac{300x^5}{y^8} - \frac{75x^4}{2y^7} + \frac{64x^9}{y^{12}} + \frac{3x^7}{y^{10}} - \frac{x^6}{8y^9} + \frac{15x^5}{4y^8} + \frac{125x^3}{y^6} + \frac{240x^7}{y^{10}} - \frac{60x^6}{y^9} - \frac{24x^8}{y^{11}}$.

72. $\frac{12}{a^7} - \frac{125}{a^{12}} - \frac{60}{a^8} + \frac{1}{a^9} - \frac{15}{a^{10}} + \frac{8}{a^6} + \frac{6}{a^8} + \frac{150}{a^{10}} - \frac{60}{a^9} + \frac{75}{a^{11}}$.

73. $\frac{135a^5}{33b^4} + \frac{125a^9}{8b^6} - \frac{9}{8b} - \frac{8b^3}{27a^6} + \frac{10}{3a} + \frac{27a^3}{64b^3} + \frac{b}{a^3} + \frac{225a^7}{16b^5} - \frac{15a^2}{2b^2} - \frac{25a^4}{2b^3}$.

74. $\frac{64a^9}{x^{12}} + \frac{171a^6}{x^9} + \frac{102a^5}{x^8} + \frac{8a^3}{x^6} + \frac{36a^4}{x^7} + \frac{204a^7}{x^{10}} + \frac{144a^8}{x^{11}}$.

75. $27b^6 + 296b^9 - 125b^{12} - 108b^7 + 9b^8 - 15b^{10} - 300b^{11}$.

Ex. 26.

Pure Quadratics.

1. $3(2x-3)^2 = 4x(2x-9)+43.$

2. $2x+\frac{17}{x} = \frac{7x-10}{2\frac{1}{2}}+4.$

3. $(x+9)(x-9) = 19-3x^2.$

4. $(x+4)(2x+9) = \left(2x+\frac{17}{2}\right)2x-13.$

5. $\frac{2x-1}{x-2} = \frac{x-5}{3x-2}.$

6. $\frac{4}{x+3}-\frac{4}{x-3} = -\frac{1}{3}.$

7. $\frac{1}{3x+11}-\frac{1}{3x-11} = -\frac{22}{23}.$

8. $\frac{1}{12-5x}+\frac{1}{5x-12} = 1+\frac{120x}{(12-5x)^2}.$

9. $\frac{5}{2+x}-\frac{5}{x-4} = \frac{2-x}{x-4}+\frac{2}{x^2-2x-8}.$

10. $\frac{3x^2}{4}-\frac{15x^2+8}{6} = 2x^2-3.$

11. $\frac{x+7}{x(x-7)}-\frac{x-7}{x(x+7)} = \frac{7}{x^2-73}.$

12. $\frac{x-3}{x^2-4}+1 = \frac{1}{x-2}.$

13. $\frac{2x}{x-2}+\frac{x-2}{x} = 2.$

14. $\frac{x+m}{x-m}+\frac{x-m}{x+m} = -\frac{4mn+2n^2}{x^2-m^2}.$

15. $\frac{x}{a^2}-\frac{b^2}{x}=\frac{x-2b^2}{a^2x}-\frac{1}{a^2}$.

16. $\frac{a^2}{x}-\frac{x}{b^2}=\frac{x}{a^2-b^2}$.

17. $(x+a)(x-a)+(x+b)(x-b)-(x+c)(x-c)=0$.

18. $(3a+b-x)(a+3b-x)=2(a+b-x)^2$.

19. $\frac{x^4-m^4}{x-m}-\frac{x^4-m^4}{x+m}=10m^3$.

21. $\frac{ab-x}{b-ax}=\frac{b-cx}{bc-x}$.

20. $\frac{n-x}{n+x}=\frac{x+p}{x-p}$.

22. $\frac{x+3a-b}{x+2a-b}=\frac{x-4a+5b}{3b-x+a}$.

23. $\frac{4a+b+x}{x+2a-b}=\frac{x-2a+b}{4a+b-x}$.

24. $\frac{17a+b-x}{a+17b-x}=\frac{a^2(a+17b+x)}{b^2(17a+b+x)}$.

25. $\frac{x^3-2bx+2ax-b^2}{x^3-a^3}+\frac{x+2b}{x^2+ax+a^2}=\frac{1}{x-a}$.

Ex. 27.

Affected Quadratics.

Numerical Quadratics of the form $x^2+px+q=0$.

1. $x^2-2x=48$.
2. $x^2-84=-8x$.
3. $10x=171-x^2$.
4. $x^2+12=7x$.
5. $14=x^2-5x$.
6. $60-x^2=-7x$.
7. $x(x-1)=380$.
8. $3x^2-5x=10+2x^2-2x$.
9. $5x^2-18x=630-9x+4x^2$.
10. $250+2x^2=3x^2-15x$.
11. $6x^2-\frac{7}{4}+4x=x+5x^2$.

12. $2-8x+3x^2=-4+2x^2-3x.$

13. $(3+5x)x-\frac{1}{9}=3x\left(1-\frac{5}{3x}\right)+x(15+4x).$

14. $(x+5)^2=2(x+3)^2-17.$

15. $(3x-2)^2=8(x+1)^2-100.$

16. $(x-5)^2-5=(2x+6)8-x^2.$

17. $(2x-8)^2=(3x+25)4+12.$

18. $\frac{x^2}{6}+\frac{2x}{3}=10.$

19. $3(2x-3)-\frac{22}{x}=\left(x-\frac{3}{2}\right)5.$

Equations with the form $ax^2+bx+c=0$:

20. $5x^2-3x-2=0.$

21. $3x^2+5x-42=0.$

22. $9x^2+15x=6.$

23. $5x^2-7x+\frac{6}{5}=0.$

24. $3x^2-50=25x.$

25. $7x^2-9x=10.$

26. $3x^2-7x-6=0.$

27. $2x^2-13x=-15.$

28. $2x^2-8=3x+12.$

29. $5x(x-2)+2=-4-x(4x-5).$

30. $3\left(x+\frac{33}{25}\right)-13x^2=3x^2.$

31. $11x^2+7x-\frac{3}{7}=4x(x+1)+1.$

32. $5x(x-2)+\frac{1}{4}=1-3x.$

33. $x(12x+0.7)-\frac{7}{11}=x(3x-0.2)+6\frac{4}{11}$.

34. $11(x-1)^2+x=2(1-x)^2+11$.

35. $3(z^2+2)^2-54=3z^4+z(5z-7)$.

36. $(1-3x)(x-6)=2(x+2)$.

37. $(2x-3)(x-2)-49\frac{2}{3}=(x-1)^2+\frac{16}{3}-2x^2$.

38. $(x-1)(1-3x)-\frac{1}{3}=\left(x-\frac{1}{3}\right)3-1$.

39. $(x+1)(2x+3)=4x^2-22$.

40. $5\left(\frac{1}{8}+x\right)^2+4x=(3\frac{7}{8}-x)^2-\frac{1}{2}$.

41. $(3x+1)(2x-3)+3=(x+3)^2-15$.

42. $\frac{x-3}{x+4}+\frac{x-4}{2(x-1)}=\frac{1}{2}$.

43. $x-3+\frac{x+6}{x-6}=2x-7$.

44. $\frac{2x-1}{3}-\frac{3}{x-8}=\frac{x-2}{x-8}+3$.

45. $\frac{3x+2}{2x-1}+\frac{7-x}{2x+1}=\frac{7x-1}{4x^2-1}+5$.

Equations with the form $ax^2+2bx+c=0$:

46. $x^2+6x-11\frac{1}{4}=0$.

47. $x^2-4x+2=-\frac{14}{25}$.

48. $3x^2-6x=-\frac{5}{3}$.

49. $9x^2+12x-5=7$.

50. $5x^2-8x+3=0$.

51. $7x^2-48=2x(x+7)$.

52. $(5x+1)^2-2\left(x+\frac{1}{5}\right)=10x^2+16\frac{3}{5}$.

53. $(5x-2)(x+1)=\left(x+\frac{3}{5}\right)5-5$.

54. $(3x+2)(x-3)=(6-x)5+4$.

55. $13x^2-30=6(1-x)^2+63$.

56. $(3x+2)(5x-2)=(5x-1)^2-17$.

57. $(1.2-x)^2+(x+0.8)^2=2(6x-0.2)^2$.

58. $(0.7+x)^2+(1.3-x)^2-(1.6-2x)^2=\frac{10x}{3}$.

59. $(3x+0.5)^2+(5x-0.5)^2=6(3x-0.5)^2+4x$.

60. $\frac{x-2}{x-1}-\frac{x-3}{x+3}=\frac{x+4}{x-1}-\frac{7}{4}$.

61. $\frac{6}{2x-6}-\frac{1}{x-3}=\frac{x-5}{4}$.

62. $\frac{x-5}{x+3}+\frac{x-8}{x-3}=\frac{80}{x^2-9}+\frac{1}{2}$.

63. $\frac{2x+1}{7-x}+\frac{4x+1}{7+x}=\frac{45}{49-x^2}+1$.

64. $(x+1)^2+(2x-1)(1-x)=(x+2)(2x-1)-9.27$.

65. $(x+0.9)^2+(x-1.1)(2x-0.2)=(3x-4.3)^2+3(x-2.1)$.

66. $\frac{x^2}{2}-\frac{x}{3}+\frac{59}{8}=8$.

67. $\frac{x^2}{4}-\frac{3x}{8}-\frac{7}{6}=1\frac{1}{3}$.

68. $\frac{x+1}{3}+\frac{3(x-1)}{4}=(x-3)^2+1$.

69. $\frac{2x+5}{5}-\frac{x-4}{3}=\frac{2x(x-7)}{6}-\frac{7x}{10}$.

70. $\frac{x-2}{6}+\frac{(x-4)(x-6)}{8}-\frac{2(16-x)(x-5)}{3}=0.$

71. $\frac{(x-8)(x-10)}{4}-\frac{(x-12)^2}{8}=\frac{x(x-15)}{5}+\frac{x}{10}.$

72. $\frac{(x-12)^2}{6}-\frac{x}{9}+\frac{x(x-9)}{18}=\frac{(x-14)^2}{2}+5.$

73. $\frac{x(2x-10)}{12}-\frac{(x-7)^2}{2}=\frac{(14-x)^2}{3}+(11-x)^2.$

74. $\frac{x-1}{3}+\frac{x-2}{7}-\frac{2x-7}{5}=\frac{x(x-15)}{8}.$

75. $\frac{(x+3)^2}{5}-\frac{(2x-20)(x-12)}{3}-\frac{7(x-6)}{2}=-\frac{x}{2}.$

76. $\frac{x-1}{x-5}=\frac{3x}{2x-7}.$

77. $\frac{3x+2}{x-2}=\frac{4x+5}{x-1}.$

78. $\frac{x-7}{2(x+3)}=\frac{x-6}{x+24}.$

79. $\frac{x+0.5}{x-0.5}=\frac{2x+1}{x+2.5}.$

80. $\frac{4x-1}{x+0.5}=\frac{6x}{x+2.5}.$

81. $\frac{3x-1}{4x+\frac{1}{3}}=\frac{2x+\frac{2}{3}}{5x+2\frac{2}{3}}.$

82. $\frac{x+1}{x-1}=\frac{4x-3}{x+9}.$

83. $\frac{2x-1}{x-4}=\frac{x+1}{x-7}.$

84. $\frac{x}{6}+\frac{35}{x+4}=4\frac{1}{2}.$

85. $\frac{55}{16-x}-\frac{x}{6}=4\frac{1}{2}.$

86. $\frac{6}{x^2-1}+\frac{3}{x+1}=\frac{2}{x-1}+1.$

87. $\frac{2(x+7)}{x+1}+\frac{x-1}{x+1}=\frac{x+11}{x^2-1}+4.$

88. $\frac{x+9}{x-3}+\frac{2(2x+1)}{x+3}=\frac{7}{2}+\frac{7(x+13)}{x^2-9}.$

89. $\frac{2x+1}{x+3}-\frac{x-1}{x^2-9}=\frac{x+3}{3-x}-\frac{4+x}{3+x}$.

90. $\frac{1}{x-3}+\frac{7}{x+3}=\frac{14}{x^2-9}-\frac{x-4}{x+3}$.

91. $\frac{x}{2x-1}+\frac{25}{4x^2-1}=\frac{13}{2x-1}+\frac{1}{27}$.

92. $\frac{15}{3x-1}-\frac{3}{4}=\frac{220}{9x^2-1}-\frac{11}{3x+1}$.

93. $\frac{8-x}{5(x^2-4)}+\frac{x+2}{x-2}+\frac{x-2}{x+2}=\frac{34}{15}$.

94. $\frac{x+4}{3x-7}-\frac{9}{19}-\frac{x}{3x+7}=\frac{52}{9x^2-49}-\frac{15}{52}$.

95. $\frac{1}{x-2}+1-\frac{6-x}{3(x^2-4)}-\frac{1}{2-x}=0$.

96. $\frac{x+1}{x-1}+\frac{x+2}{x-2}-\frac{2x+13}{x+1}=0$.

97. $\frac{x-1}{2x+3}-\frac{x+1}{2x-3}=\frac{x+7}{4x^2-9}-\frac{x-3}{2x-3}$.

98. $\frac{2x-1}{x+1}+\frac{3x-1}{x+2}-\frac{x-7}{x-1}=4$.

99. $\frac{1}{18+x}-\frac{1}{7}=\frac{1}{11}+\frac{1}{x}$.

100. $\frac{1}{24+x}-\frac{1}{13}=\frac{1}{x}+\frac{1}{11}$.

101. $\frac{3x+1}{3(x-5)}-\frac{2x-7}{2x-8}-\frac{5}{2}=0$.

102. $10x-\frac{14x-9}{8x-3}=\frac{18-40x^2}{3-4x}-9$.

103. $\frac{x^2+2x+2}{x+1}+\frac{x^2+8x+20}{x+4}=\frac{x^2+4x+6}{x+2}+\frac{x^2+6x+12}{x+3}$.

104. $\frac{2x-3}{5}+\frac{1}{7}\left(5x-\frac{6x+4}{5x+1}\right)=x+\frac{5x+8}{3x-14}+\frac{1}{3}\left(\frac{x}{7}+\frac{x-9}{5}\right)$.

105. $\frac{x+3}{x+2}+\frac{x-3}{x-2}=\frac{3x+1}{2\left(x-\frac{4}{x}\right)}-\frac{12}{x^2-4}$.

Literal equations with the form $x^2+px+q=0$:

106. $x^2+ax+b=0$.

107. $x^2-ax+b=0$.

108. $x^2+ax-b=0$.

109. $x^2-ax-b=0$.

110. $ax+b=x^2$.

111. $x^2+n=-mx$.

112. $x^2-dx=g$.

113. $bx+c=x^2$.

114. $px+4p^2=x^2-2px$.

115. $bx-2b^2+x^2=2bx$.

116. $x^2+dx-h^4=3h^4-\frac{d^2}{4}$.

117. $x^2-\frac{x}{a}=\frac{3}{4a^2}$.

118. $x^2+mn=-(m+n)x$.

119. $x^2-(n+p)x=-np$.

120. $x^2-\frac{m^2+n^2}{mn}x=-1$.

121. $x^2+\frac{3nx}{2}-n^2=\frac{11n^2}{16}$.

122. $cx-2c-x^2=-2x$.

123. $x^2-\frac{ax}{2}=\frac{a^2}{2}$.

124. $x^2+\frac{mx}{3}=\frac{2m^2}{3}$.

125. $x^2-\frac{3px}{5}=-\frac{p^2}{20}$.

126. $x^2-\frac{5bx}{3}=-\frac{b^2}{4}$.

127. $x^2-\frac{3(a^2+b^2)}{4}-ax=\frac{b}{2}(3a+2x)$.

128. $(m-1)^2x^2-m(x+n)=mx^2(m-2)-nx$.

129. $(x+a)(x-b)+ab=2x^2-ab$.

130. $(m+x)(x-n)=2(x-n)^2+mn$.

131. $\frac{x}{a}\left(1+\frac{x}{b}\right)+1+\frac{x}{b}=0$.

Literal equations with the form $ax^2 + bx + c = 0$:

132. $3x^2 + ax = b$.

133. $2x^2 - 3ax = 2a^2$.

134. $p - ax = -bx^2$.

135. $cx^2 + 5ax = 5a + c$.

136. $ax^2 + 5a^2x = -\frac{9a^2}{4}$.

137. $2x^2 - (a+b)x + \frac{ab}{2} = 0$.

138. $(x-p)^2 = 5px - (x+p)^2$.

139. $2ax^2 - (x + 5a)\,bx = -2b^2(x + a)$.

140. $(2a + x)(2a - x) = 15ab - 9b^2 - 3ax$.

141. $(x + a)(x - 2b) + ab = \frac{3(a+b)^2}{4} - bx$.

142. $(x^2 + px + p^2)(x + p) = (x^3 + p^3)(x - 2p) + 6p^3$.

143. $acx^2 - bcx + adx - bd = 0$.

144. $\frac{bx}{c}(acx + b) + \frac{3a^2x}{c} = \frac{6a^2 + ab - 2b^2}{c^2}$.

145. $(a - b)x^2 + \frac{ab}{a-b} = (a + b)x$.

146. $abx^2 = \frac{1}{ab}\left[x(a + b) - \frac{1}{ab}\right]$.

147. $\frac{3a}{x+a} + \frac{2a}{x+2a} - \frac{a}{x+3a} = \frac{4a}{x}$.

Literal equations with the form $ax^2 + 2bx + c = 0$:

148. $ax^2 - 2bx = c$.

149. $ax^2 + 2bx = -c$.

150. $px^2 - 4qx = m$.

151. $-mx^2 - c = 4mx$.

152. $c^2x^2 = 8cx + a^2 + 8a$.

153. $35a^2 + 3x^2 = 22ax$.

154. $21a^2b^2 - 44abx = -15x^2$.

155. $(a - x^2)\,b = (b - 1)\,x^2$.

156. $(m - x)(n + x) = x(n - m + 2x) - \frac{mn}{3}$.

157. $(a + x)^2\,b = (a + b)^2 - x^2 - \frac{b^2(2a+b)}{b+1}$.

158. $(x+a)(3x-b)=(3b-a)x-(a-b)^2-ab.$

159. $x^2-4bx=a^2-2ab-3b^2.$

160. $ax\left(\frac{ax}{b^2}-\frac{1}{c}\right)+\frac{1}{c}\left(\frac{b^3}{c}-ax\right)=0.$

161. $\frac{4b}{a}+\frac{a-4b}{x-2b}=\frac{a+4b}{x+2b}.$

162. $\frac{1}{x}+\frac{1}{x-2a^2}-\frac{1}{b^2}=0.$

163. $\frac{1}{x+a}+\frac{1}{x+b}-\frac{1}{x+c}=0.$

164. $\frac{1}{a}+\frac{1}{a+x}+\frac{1}{a+2x}=0.$

165. $\frac{a}{x-a}-\frac{2a}{x-2a}+\frac{3a}{x-3a}=0.$

166. $\frac{x}{a-x}-\frac{a-x}{x}=\frac{a}{x}.$

Find the best solution for the following equations:

167. $abcx^2-a^2b^2x+c^4x-abc^3=0.$

168. $abx^2-(a^2-b^2)x-ab=0.$

169. $x(x-5a)-b(x+6b)=-a(6a+5b).$

170. $4b(b-x)=(a+x)(a-x).$

171. $b(a-x)-c(a+x)=x(a-x)-2c(b-c).$

172. $x(x-2b)+a(b-x)+2c(a-2c)=-b^2.$

173. $x^2-(a^2+b^2)x=2ab(x-a^2-b^2).$

174. $ax(cx-3b)=5d(3b-cx).$

175. $a^5x(c^4x-a^3b)=(a^3b-c^4x)b^2c^3.$

176. $p^3x(m^6p^2x-n^4)+n^2x(m^3-n^2p^3)=n^2\left(\frac{4n^4}{m^3p^2}-m^3x\right).$

177. $x(x+b^2-b)=ax(a+1)-(a+b)^2(a-b).$

178. $x^2+(4m^2-n^2)2mn=(4m^2+n^2)x.$

179. $\frac{a}{x-a}+\frac{b}{x-b}=\frac{2c}{x-c}.$

180. $\frac{1}{x}+\frac{1}{x+d}=\frac{1}{c}+\frac{1}{c+d}.$

181. $\frac{p^2+x^2}{p^2-x^2}=\frac{p+2x}{p-2x}.$

182. $\frac{ax^2-bx+c}{mx^2-nx+p}=\frac{c}{p}.$

Ex. 28.

Problems Involving Quadratics.

1. Two numbers are in the ratio of $\frac{1}{3}$ to $\frac{1}{5}$, and the sum of their squares is 1666. What are the numbers?

2. The sum of the squares of three integral consecutive numbers is 1202. What are the numbers?

3. The sum of the squares of three consecutive even numbers is 776. What are the numbers?

4. The sum of the squares of three consecutive multiples of 4 is 3104. Find the numbers.

5. In the place of a square field which I have bought, I am offered a rectangular piece of ground of the same perimeter, but the length is 16 rods more than that of the square, and the area is only $\frac{224}{225}$ of that of the square. What is the value of the rectangular garden at \$4 the square rod?

6. The sum of the squares of two numbers is 5690, and the difference of the squares is 1272. What are these numbers?

7. A circular space covered with turf has a surface of 1017.8784 square meters. It is required, without changing the surface, to give to the turf the form of a ring included between two concentric circumferences, the radius of one being double that of the other. What radii must be given to these circumferences? The ratio of the circumference to the diameter is 3.1416.

8. The length of a rectangular field is to its width as 16 is to 11. A neighbor, to make more regular the form of his adjoining property, proposes to diminish the width, but to increase the length so that the new dimensions will be in the ratio of 17 to 10. The rectangular field thus loses 216 square yards. What are its dimensions?

9. Two travellers set out at the same instant from the same point and go one to the south and the other to the east. The first accomplishes 30 miles and the second 40 miles per day. In how many days will they be 250 miles distant from each other?

10. A man buys a certain number of kilograms of coffee, 4 times as much of tea, and 6 times as much of wool. He pays per kilogram for each, as many francs as the number of kilograms he has bought of each. He pays for all 477 francs. How much coffee did he buy?

11. A traveller sets out from A and goes toward the north, accomplishing each day 36 miles. Five days after, a second traveller sets out from A going toward the east, and making each day 28 miles. At the end of how many days will the second traveller be 430 miles from the first?

12. The sum of the squares of three consecutive integral numbers, diminished by the sum of the squares of the two integral numbers preceding, is 288. Find the numbers.

13. A certain even number is such that its square, increased by those of the two even numbers following, is greater by 406 than the sum of the squares of the two odd numbers which follow it. Find the number.

14. Of two numbers, the greater is 48 more than the smaller, and if one adds to their sum the quotient of the greater divided by the smaller, 99 is the result. Find the numbers.

15. A number is the product of three integral consecutive numbers. If it is divided by each of its three factors, the sum of the quotients thus obtained is 767. What is this number?

16. Find two numbers whose sum is double their difference, and whose product diminished by the smaller is equal to 720 times the quotient of the greater divided by the smaller.

17. Find a number which, added to its square and cube, has for the sum 463 times the number in question.

18. A circular piece of water of 864 square yards is changed to another of the same extent but of rectangular form, whose width will be 12 yards less than the length. What will be the two dimensions?

19. The surface of a rectangular field is 312.8^{a}. If each of the dimensions are increased 1^{m}, the surface will be increased 472^{qm}. What are the two dimensions?

20. If 5 pounds of beef more are sold for \$2, twelve pounds will cost \$0.40 less. How much does one pound cost?

21. A woman buys a certain number of books for \$1.80. If for the same price she had received 4 books more, each would have cost $1\frac{1}{2}$ cents less. How many books did she buy?

22. A merchant bought cattle which cost him in all \$672. If each head had cost him \$4 less, he would have been able to buy 3 more. How many did he buy, and at what price?

23. A man buys 673 young trees which he plants in rows, equally distant, upon two pieces of land forming squares. The side of one contains 6 trees more than the side of the other, and after this he has 7 trees left. How many trees does each square contain?

24. A father is 24 years older than his son. The product of the ages of the father and son is equal to 3 times the square of the age of the son. What is the age of the father and of the son?

25. A tourist, who has accomplished a journey of 630 miles, would have taken 4 days less if he had travelled each day 10 miles more. How many days did it take him to accomplish his journey, and how much did he travel each day?

26. A tourist, who has accomplished a journey of 630 miles, would have taken 4 days more if he had travelled each day 10 miles less. How many days did it take him to accomplish his journey, and how much did he travel each day?

27. A young merchant buys a certain number of melons which cost him $7.56. After throwing away 5 bad ones, he sold each one which remained for 4 cents more than he had paid for it, and thus gained $0.58. How many melons did he buy?

28. Eight hundred and forty nuts were distributed among some children. If each one had received 2 nuts less he would have had as many as there were sharers in the distribution. How many children were there?

29. I sell a watch for $22.75, and I thus lose as much per cent as it had cost me. How much did I pay for it?

30. The side of a square is 230 yards. Find the dimensions of a rectangle whose perimeter is 12 yards more than that of the square, and the area 460 square yards less.

31. A square field contains a number of square yards equal to the product of 32 times its perimeter, increased by 260. What is the side of the square?

32. A square piece of land, in order to have an area of 260 square yards, must be increased by a number of square yards equal to 32 times the perimeter of the square. What is the side of the square?

33. A woman distributes 144 nuts among 7 children, giving half to the girls, and half to the boys. In this manner each girl receives 6 nuts more than a boy. How many boys and girls were there?

34. Two reservoirs of a cubic form contain together 1853^{l}. The sum of the heights of the reservoirs is 1.7^{m}. What is the side of each of them?

35. Three workmen, A, B, C, dig a ditch; A in 6 days more, B in 30 days more than the number of days it takes them to do it together; and C in 3 times the number of days it takes them to do it together. How many days would it take them to dig the ditch if they all work together?

36. I buy oranges, each of which costs a number of cents equal to one-third of the number of oranges bought. With the same sum I could buy twice as many melons less 21, paying for each 12 cents more than for an orange. How many oranges did I buy?

37. The denominator of a fraction is greater than the numerator by 3. If 2 is added to each term, the fraction thus obtained is $\frac{2}{21}$ more than the first fraction What is the first fraction?

38. A cask contains 80^l of wine. A person draws off a certain number of liters, which he replaces by water. From this mixture he again draws off the same number of liters as the first time, and again replaces them by water, after which the 80^l of the cask includes only 45^l of wine. How many liters did he draw off each time?

39. Two messengers, starting from the same point, are sent at the same moment to a city at a distance of 72 miles. The first accomplishes 1 mile per hour more than the second, and arrives at his destination 1 hour sooner. What distance does each messenger travel each hour?

40. A train starts from A for B, a distance of 216 miles, at 2 o'clock in the morning. At 5.30 o'clock there starts from A; for the same destination, a second train which travels 4 miles more per hour than the first, and arrives 1 hour sooner. At what hour does each train arrive in B?

41. A tank can be filled in 26$\frac{1}{4}$ hours by two pipes together. If they work separately, the tank will be filled in 18 hours less by the larger than by the smaller. How many hours will it take each pipe to fill the tank?

42. Two numbers are in the ratio m to n; the sum of their squares is s. Find the numbers.

43. Let a, b, c be the three sides of a triangle, and a the greatest. Find the quantity x that must be taken away from each side in order that the triangle whose sides will be $a-x$, $b-x$, $c-x$, will be a right triangle.

Apply the result to the triangle whose sides are 18, 16, and 9.

44. Divide a line a in 2 segments such that twice the sum of the squares made upon the segments equals the square made upon the entire line.

45. Some friends, dining together at a hotel, incur a total expense of \$102. Three of them, being guests, were not allowed to pay; and the others paid \$1$\frac{1}{7}$ more than if each had paid his share. How many friends were there in all?

46. If I take the product of $\frac{1}{80}$, $\frac{1}{20}$, $\frac{1}{16}$ of a certain number, the product is 16 times this number. Find the number.

47. What is the number which is greater than its square root by 182?

48. Two trains set out from two cities, M and N, distant 560 miles, to meet each other. To meet at half the distance, the train from N must start $1\frac{3}{4}$ hours before the other. If the two trains start at the same moment, at the end of 7 hours the distance which separates them will be $\frac{1}{10}$ of the first distance. How long does it take each train to go from M to N?

49. Two bodies move upon 2 lines which form a right angle, and approach the point of intersection. One body is at a distance of 236 yards, and passes over 7 yards per second; the other is distant 197 yards, and passes over 6 yards per second. At the end of what time will they be 13 yards distant from each other?

Ex. 29.

Roots of Quadratic Equations.

If in the equation $x^2 + px + q = 0$, α and β are the roots; then

$\alpha + \beta = -p$, $\alpha\beta = q$, and $(x-\alpha)(x-\beta) = x^2 + px + q$.

If the equation has the form $ax^2 + bx + c = 0$,

$\alpha + \beta = -\frac{b}{a}$; and $\alpha\beta = \frac{c}{a}$.

Form an equation of the second degree whose roots are the following:

1. 3 and 5.
2. 7 and 3.
3. 1 and 2.
4. 8 and 17.
5. 3 and -1.
6. -2 and 7.
7. 5 and -6.
8. -9 and 4.
9. $\frac{1}{2}$ and $\frac{2}{3}$.
10. $\frac{3}{4}$ and $\frac{5}{6}$.
11. 12 and $\frac{7}{8}$.
12. 4 and $-\frac{3}{5}$.
13. -7 and $\frac{3}{5}$.
14. -2 and $-\frac{1}{5}$.
15. 1 and -1.
16. 11 and -11.
17. -6 and -4.
18. -2 and -10.
19. -5 and -11.
20. -3 and $-\frac{1}{4}$.
21. 4.3 and 2.5.
22. 0.4 and 2.5.
23. -0.7 and 0.3.
24. -8.2 and 5.1.
25. -2.4 and -1.3.
26. 2 and 0.5.
27. 4 and 0.25.
28. -7 and $-\frac{1}{7}$.
29. 5 and $-\frac{1}{5}$.
30. -3 and $\frac{1}{3}$.
31. a and b.
32. $a+b$ and $a-b$.
33. a and a^2.
34. $a+3b$; $3a+b$.
35. a and $-a$.
36. a and $-b$.
37. $-m$; $-n$.
38. $3m$; $-2m$.
39. m; 2.
40. $\frac{a}{2}$; $\frac{1}{3}$

If $x^2+px+q=0$; then

$$x=\frac{-p\pm\sqrt{p^2-4q}}{2}.$$

41. In what cases are the roots real?
42. When are they imaginary?
43. Being real, when are they unequal?
44. When are they equal?
45. When are both of the roots positive?
46. When are both negative?
47. In what case is the greatest in absolute value positive, and the least negative?
48. When is the smallest positive and the largest negative?
49. When are the roots commensurable?
50. When are they irrational?

Are the roots of the following equations real or imaginary, positive or negative? Answer without solving:

51. $x^2-6x+8=0.$
52. $x^2-5x+6=0.$
53. $x^2-x-12=0.$
54. $x^2+12x+11=0.$
55. $x^2+9x+8=0.$
56. $x^2+15x+14=0.$
57. $4x^2+52x=87.$
58. $x^2+25x-26=0.$
59. $x^2+11x-12=0.$
60. $8x+20=x^2.$
61. $x^2-21x-46=0.$
62. $16x-63=x^2.$
63. $x^2-16x+64=0.$
64. $x^2=18x-81.$
65. $x^2+10x+25=0.$
66. $2x^2-18x+65=0.$
67. $x^2-31x+246\frac{1}{2}=0.$
68. $2x+5=x^2.$
69. $x^2=2-14x.$
70. $x^2+6x+4=0.$
71. $x^2+8x+25=0.$

72. If the roots of the equation $ax^2+bx+c=0$, are denoted by α and β, what is the equation whose roots are $\frac{1}{\alpha}$ and $\frac{1}{\beta}$?

Ex. 30.

Maxima and Minima.

If x may have any value, positive or negative, find the values of x which give to the following expressions a maximum or minimum value:

1. $16-x^2$.
2. $(3+x)(3-x)$.
3. $x(8-x)$.
4. $x(x-10)$.
5. x^2-6x+5.
6. $2x^2+x-3\frac{7}{8}$.
7. $3x^2-4x-\frac{8}{3}$.
8. $(x-1)^2-3$.
9. $(4x-7)^2-8x$.
10. $12x-(3x-1)^2$.
11. $x^3+(4-x)^3-2$.
12. $(x-4)^3-x^3+20$.
13. $\frac{x-6}{x^2}$.
14. $\frac{x-1}{x^2}$.
15. $\frac{(x+12)(x-3)}{x^2}$.
16. $\frac{3+x}{3-x}+\frac{3-x}{3+x}$.
17. $\frac{1}{2+x}+\frac{1}{2-x}$.
18. $\frac{2x}{1+x^2}$.
19. $\frac{x^2+2x-3}{x^2-2x+3}$.
20. $\frac{4x}{x^2+4x+4}$.
21. $\frac{m^2x^2+n^2}{(m^2-n^2)x}$.
22. $\frac{64x^2+36}{28x}$.
23. $\frac{(x+1)^2}{x^2-x+1}$.
24. $\frac{x^2-1}{x(x+1)}$.
25. $\frac{x^2-x-1}{x^2-x+1}$.

26. $\frac{25}{x}+x$.

27. $x+\frac{16}{x}$.

28. $\frac{(20+x)(5+x)}{x}$.

29. $\frac{2x^2+3x-5\frac{7}{8}}{x-3}$.

30. $\frac{2x^2-2x+5}{x^2-2x+3}$.

31. $\frac{4(x+2)}{4x^2+8x+9}$.

32. $\frac{x^2+4x+1}{x^2-4x+1}$.

33. $\frac{x^2+3x+5}{x^2+1}$.

34. $\frac{x-4}{x^2-3x-3}$.

35. $\frac{x}{a^2+x^2}$.

Find the maximum or minimum values in the following examples, if x can have positive values only:

36. $\frac{(x+1)^2}{x}$.

37. $\frac{x^2}{x-1}$.

38. $\frac{x^2-2x+1}{x-3}$.

39. $\frac{x^2-2x+2}{2(x-1)}$.

40. $\frac{x}{(x-3)(x-2)}$.

41. $\frac{x^2-4x+3}{x^2-1}$.

42. $x^3-x^2-16x+10$.

43. x^3-2x^2-x+3.

44. $2x^3-5x^2-4x+19$.

45. $\frac{x^3}{3}-\frac{3x^2}{2}+2x-21$.

Study the variation of the following functions:

46. $y=x^2-6x+5$.

47. $y=x^4-8x^2-9$.

48. $y=x+\frac{1}{x}$.

49. $y=\frac{x^2-3}{2-x}$.

50. $y=\frac{x^2-5x+1}{x^2-x+1}$.

51. $y=\frac{x+4}{7-x^2}$.

52. A given number $2a$ is to be divided into two parts whose product is maximum. Find these parts.

53. Of all the rectangles whose perimeter is $4a$, what is the one whose surface is maximum?

54. Divide a number $2a$ into two such parts that the sum of their square roots is a maximum.

55. Divide a number $2a$ into two such parts that the sum of their squares is a minimum.

56. Of all the triangles of the same perimeter, what is that one whose surface is the greatest?

57. What is the greatest of all the rectangles that can be inscribed in a circle?

58. Inscribe in a triangle a rectangle whose surface is a maximum.

59. Inscribe in a cone, the radius of whose base is r and the height h, a cylinder whose volume is a maximum.

60. Inscribe in a given square whose side is a, another square whose area is a minimum.

61. Circumscribe about a cylinder whose height is h and the radius of the base r, a cone of minimum volume.

62. Of all the cylinders that can be inscribed in a sphere, what is that one whose lateral surface is a maximum?

63. Of all the isosceles triangles that can be inscribed in a circle, what is that one whose surface is a maximum?

64. Of all the cones circumscribed about a sphere, what is that one whose volume is a maximum?

65. Of all the cones circumscribed about a sphere, what is that one whose volume is a minimum?

Ex. 31.

SIMULTANEOUS QUADRATICS.

1. $x^2+y^2=661$,
$x^2-y^2=589$.

2. $y^2-z^2=-80$,
$z^2+y^2=82$.

3. $3x^2-y^2=59$,
$2x^2+3y^2=98$.

4. $\frac{x^2}{2}+3y^2=140$,
$5x^2-\frac{y^2}{3}=308$.

5. $x+y=12$,
$xy=35$.

6. $y-z=8$,
$yz=240$.

7. $x+y=\frac{7}{6}$,
$xy=\frac{1}{6}$.

8. $x+y=1$,
$xy=-1$.

9. $x^2+y^2=74$,
$x+y=12$.

10. $\frac{x+y}{x-y}=12$,
$x^2-y^2=48$.

11. $\frac{1}{x}-\frac{1}{y}=\frac{1}{36}$,
$xy^4-x^2y=324$.

12. $5x^2+2y^2=373$,
$2x+5y=54$.

13. $16x^2=9y^2$,
$2xy-5x+6y=33$.

14. $60-16xy=-x^2y^2$,
$x+y=7$.

15. $y^2+z^2=130$,
$y+z=14$.

16. $x^2+z^2=\frac{241}{400}$,
$x+z=\frac{19}{20}$.

17. $x^2+y^2=250$,
$x-y=22$.

18. $x^2-y^2=120$,
$x+y=20$.

19. $x^2-y^2=-\frac{1}{4}$,
$x+y=\frac{5}{2}$.

20. $x^2+xy=260$,
$xy+y^2=140$.

21. $x^2+y^2=218$,
$xy-y^2=42$.

22. $5x+2y=29$,
$5xy=-105$.

23. $\frac{9}{x^2}-2y^2=-1$,
$\frac{2}{x^2}+9y^2=\frac{83}{9}$.

24. $5x^2 - 4y^2 = 109,$
$7x - 5y = 25.$

25. $x + xy + y = 47,$
$x + y = 12.$

26. $x^2 + xy + y^2 = 217,$
$x + y = 17.$

27. $x + y = 23,$
$x + xy = 144.$

28. $xy + x = 20,$
$xy - y = 12.$

29. $xy + x = 104,$
$xy - y = 84.$

30. $2x + 3y = 20,$
$3xy - y^2 = 38.$

31. $\frac{1}{x} + \frac{1}{y} = \frac{9}{20},$
$\frac{1}{x^2} + \frac{1}{y^2} = \frac{41}{400}.$

32. $\frac{1}{x} + \frac{1}{y} = \frac{1}{2},$
$\frac{1}{x^2} + \frac{1}{y^2} = \frac{5}{36}.$

33. $\frac{1}{x} - \frac{1}{y} = \frac{7}{60},$
$\frac{1}{x^2} + \frac{1}{y^2} = \frac{169}{3600}.$

34. $x^2 + xy + y^2 = 79,$
$x^2 - xy + y^2 = 37.$

35. $xy - x^2 - y^2 = -91,$
$x^2 + y^2 + xy = 223.$

36. $42(x^2 + y^2) = 85xy,$
$15xy = 2520.$

37. $x^2 + y^2 + xy = 67,$
$x + y = 9.$

38. $x^3 + y^3 - xy = 14\frac{1}{4},$
$x + y = 4\frac{1}{2}.$

39. $x^2 + y^2 + xy = 54\frac{1}{9},$
$x - y = 6\frac{1}{3}.$

40. $x^3 + y^3 - xy = 84,$
$x - y = 2.$

41. $x + y + x^2 + y^2 = 162,$
$x - y + x^2 - y^2 = -102.$

42. $x + y + x^2 + y^2 = 1\frac{5}{8},$
$y - x + y^2 - x^2 = -1.$

43. $x^2 + y^2 + x + y = 18,$
$2xy = 12.$

44. $x^3 + y^3 = 65,$
$x + y = 5.$

45. $x^3 - y^3 = 1304,$
$x - y = 8.$

46. $x^3 + y^3 = 152,$
$x^2 - xy + y^2 = 19.$

47. $x^3 - y^3 = 279,$
$x^2 + xy + y^2 = 93.$

48. $x^3+y^3=280,$
$x^2-xy+y^2=28.$

49. $x^2-xy+y^2=7,$
$3x^2+13xy+8y^2=162.$

50. $x^3+y^3=152,$
$x^2y+xy^2=120.$

51. $x^3-y^3=335,$
$xy^2-x^2y=-70.$

52. $x^3+y^3=855,$
$xy(x+y)=840.$

53. $x^3-y^3=602,$
$xy(y-x)=-198.$

54. $x^2y^4+y^2=17,$
$xy^2+y=5.$

55. $(x^3+y^3)(x+y)=671,$
$(x^3-y^3)(x-y)=11.$

56. $(x+y)(x^3+y^3)=272,$
$(x^3-y^3)(x-y)=32.$

57. $(x+y)(x^3+y^3)=1695,$
$(x^3-y^3)(x-y)=15.$

58. $(x+y)(x^3+y^3)=175,$
$(x^3-y^3)(x-y)=7.$

59. $(x^2+y^2)(x^4-y^4)=\frac{800}{9}x^2y^2,$
$(x^2+y^2)(x+y)=\frac{40}{3}xy.$

60. $(x^2-y^2)(x+y)=200,$
$x^2-y^2+x+y=30.$

61. $\frac{x^3-y^3}{x^2-y^2}=7,$
$\frac{x^2+y^2}{x+y}=10.$

62. $xy\div z=8,$
$xz\div y=18,$
$yz\div x=4\frac{1}{2}.$

63. $x(y+z)=90,$
$y(x+z)=84,$
$z(x+y)=48.$

64. $(2+x)(z-3)=30,$
$(5-y)z=6,$
$xy=32.$

65. $xy+3xz-5yz=37,$
$2xy-6xz-7yz=50,$
$5xy-13xz-20yz=121.$

66. $x^3+y^3+z^3=\frac{481}{36},$
$x+y+z=\frac{37}{6},$
$xy=z^2.$

67. $xy=80,$
$\frac{x}{y}=5.$

68. $2x - 5y = -18$,
$3xy = 264$.

69. $3x^2 - y^2 = 83$,
$x + y = 15$.

70. $u + x = 5$,
$y + z = 9$,
$u + y^2 = 28$,
$x + z^2 = 18$.

71. $x + y = 6$,
$u + z = 10$,
$x^2 + z^2 = 61$,
$u^2 + y^2 = 17$.

Ex. 32.

PROBLEMS INVOLVING SIMULTANEOUS QUADRATICS.

1. A shepherd, who is paid so much per head for his sheep, receives for the season $320. If he had had 10 sheep less, and had been paid 50 cents more per head, he would have received $354. How many sheep were there in his flock, and how much was he paid per head?

2. A shepherd, who is paid so much per head for his sheep, has received for the season $320. If he had had 10 sheep more, and had been paid 50 cents less per head, he would have received $354. How many sheep were there in his flock, and how much was he paid per head?

3. A woman carries to market cabbages for which she receives $4.20. If she had had 12 cabbages more, and had sold each one at 2 cents more, she would have brought back from market $2.28 more than she really received. How many cabbages did she have, and what was the selling price per dozen?

4. A gardener counts his melons, and thinks he can sell them for $6. In gathering them he finds 4 more than he had first counted, and he sells them at 15 cents more per dozen than he had expected. He receives $7.15. How many melons were there, and what was the selling price of each?

5. A woman spends $480 to buy black cloth, and $360 to buy blue cloth, and buys in all 90 yards of cloth. If for the same money she had bought cloth of which the price per yard was equal to the mean price of the two first, she would have received $13\frac{7}{11}$ yards less. How many yards did she buy of black cloth and of blue cloth, and what was the price of each?

6. In accomplishing a distance of 2730^{m}, a fore-wheel of a carriage makes 392 revolutions more than a hind-wheel. If the circumference of each wheel is increased by 30^{cm}, a fore-wheel in passing over the same distance will make only 325 revolutions more than a hind-wheel. What is the circumference of each wheel?

7. A fortress, besieged by the enemy, has still food enough to last 8 days. If 120 men leave, and each one remaining receives $\frac{4}{5}$ of a kilogram of bread less per day, the garrison can hold out 16 days. It will also have food for the same time if 200 men leave, and each soldier remaining receives $\frac{2}{5}$ of a kilogram of bread less than the first ration. How many men does the garrison contain, and what is a man's ration?

8. A garrison has still bread enough to last 11 days. It receives a reinforcement of 200 men; if they wish to make the provisions last as before, each man's ration of bread must be diminished $\frac{1}{4}$ of a pound. If 300 men leave, the provisions will last 12 days, giving to each man $\frac{1}{4}$ of a pound more per day. What is the force of the garrison, and what is the daily ration?

9. A detachment of workmen carry a pile of rocks to a certain distance in 165 hours. If there had been 8 workmen more, and each had carried 2^{kg} less, the

rocks would have been removed in 140 hours. But if, instead, the first detachment had been diminished by 8 men, and each one had carried 4^{kg} more, it would have taken 198 hours to accomplish the work. Of how many men was the detachment composed, and how much did each man carry?

10. The cities A and B are connected by a railroad bordering upon a carriage road. A carriage starts from A at the moment when a train leaves B. After having passed each other it takes the carriage 36 hours to arrive in B, and the train reaches A 9 hours after the meeting. How many hours did it take the carriage and the train to accomplish the distance AB?

11. A sets out from Constance for Berne at the moment when B sets out from Berne for Constance. When they meet A has travelled 30^{km} more than B, and continuing to walk at the same pace he arrives in Berne in 4 days, but it takes B 9 days to reach Constance. What is the distance from Berne to Constance?

12. The diagonal of a rectangle is 89^{m}. If each side of the rectangle were 3^{m} less the diagonal would be 85^{m}. What is the length of each side?

13. The diagonal of a rectangle is 65 feet. If 9 feet are added to the width of the rectangle, and 3 feet are taken away from the length, the diagonal is the same. What are the dimensions of the rectangle?

14. The hypotenuse of a right triangle is 35 feet. If the smaller side of the right angle is 5 feet less, and the greater side 2 feet more, the hypotenuse will be just 1 foot less. What are the sides of the right angle?

15. Two square gardens have a total surface of 2137 square yards. A rectangular piece of land whose dimensions are respectively equal to the sides of the two squares, will have 1093 square yards less than the two gardens united. What are the sides of the two squares?

16. The height of a trapezoid is 18 feet. Its surface is equal to that of a rectangle constructed upon the two parallel bases. Three times the smaller base added to the larger base is four times the height of the trapezoid. What are the two bases?

17. A man buys a watch, a cane, an umbrella, and a ring. The two first cost together $24, and the two last $21. If the price of the watch is multiplied by that of the cane, the product will be the same as the product of the cost of the other two articles. The sum of the squares of the prices paid is $585. How much did each article cost?

18. Four travellers, A, B, C, D, meet and reckon up their expenses. A and B together have spent $102, C and D, $96. The product of the sums spent by C and D is $2108. The sum of the squares of the amounts spent by each is $10,300. What is the amount spent by each traveller?

19. If from a number of four figures the number obtained by writing the figures in reverse order is subtracted, the remainder is 4725. The product of the figures is 672; the product of the two middle figures is 28, and the thousands' figure is 5 greater than that of the units'. What is the number?

20. The product of the sum and difference of two numbers is a; the quotient of the sum divided by the difference is b. What are the two numbers?

Ex. 33.

Indeterminate Equations.

1. $x+y=12$.
2. $11x+2y=83$.
3. $6x+31y=147$.
4. $121x+44y=286$.
5. $69x+184y=1173$.
6. $8x+13y=413$.
7. $635x+381y=54{,}737$.
8. $2504x+2817y=157{,}752$.
9. $56x+91y=945$.
10. $4x+9y=53$.
11. $\frac{3}{4}x+5y=92$.
12. $\frac{3x}{4}+\frac{y}{4}=27$.
13. $\frac{5x}{7}+\frac{3y}{7}=15$.
14. $\frac{4x}{5}+\frac{2y}{5}=18$.
15. $\frac{x+15y}{x-21}=-20$.
16. $\frac{13y-62x}{3x-12}=-26$.
17. $\frac{12x^2+3y-70}{5-3x}=-4x$.
18. $\frac{10x^2-18xy-19y^2+96y}{2x-4y}=5x+3y$.
19. $15x-47y=1$.
20. $\frac{14x-8y+1}{5}=\frac{3-2x-5y}{6}$.
21. Find two numbers which, multiplied respectively by 7 and 17, have for their products a sum equal to 1135.
22. If two numbers are multiplied respectively by 8 and 17, the difference of their products is 10. What are the numbers?
23. If two numbers are multiplied respectively by 7 and 15, the first product is 12 greater than the second. Find the numbers.

24. Find two numbers which are respectively multiples of 4 and 5, and the first of which is 6 smaller than the other.

25. Divide 89 in two parts, one of which is divisible by 3, and the other by 8.

26. Divide 314 in two parts, one of which is a multiple of 11, and the other a multiple of 13.

27. Divide 118 in two parts, the first of which is divisible by 7, and the second by 11.

28. What is the smallest number which, divided by 5 and by 7, gives each time 4 for a remainder?

29. A number divided by 7 has 1 for a remainder, and divided by 8, it has 7 for a remainder. Find the number.

30. The difference of two numbers is 151. The first divided by 8 has 5 for a remainder, and 4 must be added to the second to make it divisible by 11. What are the numbers?

31. Divide 352 in two such parts, that the first diminished by 2 is divisible by 7; and the second, increased by 5, is a multiple of 9.

32. Find pairs of fractions whose denominators are 24 and 16, and whose sum is $\frac{19}{24}$.

33. Find pairs of fractions whose denominators are 11 and 13, and whose difference is $\frac{82}{143}$.

34. How can one pay a sum of $87, giving only bills of $5 and $2?

35. A man buys calves at $5 apiece, and pigs for $3 apiece. He spends in all $114. How many did he buy of each?

36. A cook spends $72 to buy lambs at $5.80, and geese at $1.40 each. How many did he buy of each?

37. The diameter of a 5-franc piece of silver is 37^{mm}; that of a piece of 1 franc is 23^{mm}. How can one obtain the length of a meter in pieces of 5 francs and 1 franc?

38. A sovereign is worth 25.21 francs. How can one pay a debt of 317 francs with sovereigns to a person who has only 20-centime pieces to return?

39. How can a person weigh 1^{kg} of goods with weights of 30^{g} and $2\frac{1}{2}^{g}$?

40. A society made an excursion which cost for each grown person $7.15, and for each child $3.10. The total expense was $97.25. How many grown people and how many children were there?

41. A basket contains more than 300 and less than 400 pears. If they are divided among some children, giving 13 to each, 9 are left; but if 15 are given to each, 4 are left. How many pears does the basket contain?

42. A person buys silk at $2.50 a yard, and velvet at $3.50; in all he spends $106. How many yards did he buy of each?

43. Two brothers have worked, the first 18 days, and the second 5 days less; they receive together $142. How much did each earn per day?

44. A woman sells peaches at $0.15 apiece, and apples at $0.59 a dozen, and receives in all $10.87. How many peaches and apples did she sell?

45. A man asked his friend his age on his birthday, Feb. 5, 1880. The friend replied, "I was born in this century, and if you write the year of my birth and then add the four figures, the sum is just the age I am to-day." What is his age?

46. The age of my grandfather in 1808 was equal to the sum of the four figures of the year of his birth. How old was he then, and in what year was he born?

47. A servant receives \$37 to buy potatoes at \$3 a barrel, apples at \$5, and squashes at \$0.80. How many barrels can he buy of each kind?

48. What numbers divided by 2, 3, 7, have for their remainders, respectively, 1, 2, 5?

49. What numbers, divided by 7, 8, 9, have for their remainders, respectively, 6, 7, 8?

50. A farmer sells 15 calves, 14 lambs, and 13 pigs, and receives \$200. Some days after, at the same price, he sells 7 calves, 11 lambs, and 16 pigs, for which he receives \$141. What is the price of each?

51. Divide 111 in three parts, of which the first is divisible by 2, the second by 5, and the third by 7, and three times the first part, two times the second, and four times the third is equal to 400.

Ex. 34.

Inequalities.

Add the following inequalities, member to member:

1. $5 > 3$; $8 > 2$.
2. $a + 1 < x^2$; $a - 5 < 2x$.
3. $x^2 + y > 3 - a$; $y^2 - 2a > b + 4$.
4. $3x + y < 2a - 1$; $3y - 2x < 14 - 2a$.

Subtract the second inequality from the first:

5. $15 > 12$; $3 < 7$.

6. $6\frac{1}{2} < 10$; $2\frac{1}{2} > 1$.

7. $a^2 < 3 - x$; $2x > b^2$.

8. $a(a - x^2) > b^2 - 1$; $a^2 - 2ax^2 < 7$.

Find the products of:

9. $3 > -1$ by 7.

10. $3 > 0.5$ by -2.

11. $4a > -x$ by -2.

12. $(a-b)^2 > (x+y)^2$ by -1.

13. $18 - y^3 < 5$ by a^2.

Find the quotients of:

14. $9 > 6$ by 3.

15. $147 > 91$ by 7.

16. $a^2 - b^2 > a - b$ by $a - b$.

17. $13x^2 + 26b > 91x^2$ by -13.

Perform the operations indicated:

18. $(a - 1 < 7)(a + 1 < 10)$.

19. $(a + b > m)(n < a - b)$.

20. $(a^2 + ab + b^2 > x - y)(x + y < a - b)$.

21. $(8 > 2) \div (1\frac{1}{2} < 7)$.

22. $(x^2 - y^2 > a^2) \div (2a > x + y)$.

23. $(30 > 12)^2$.

24. $(a + b > a - x)^3$.

25. $(x + 1 < y)^4$.

26. $(a - 1 < b - 2)^5$.

27. $(-1 > -2)^5$.

28. $(3 - e > -1)^7$.

29. $[-(1 + a) < -b]^4$.

30. $[-a(1 + m) > x]^3$.

31. $\sqrt[3]{27 > 8}$.

32. $\sqrt[3]{-125 < +64}$.

33. $\sqrt[3]{729 > 343}$.

34. $\sqrt{36 > 25}$.

In exercises like the 23d, each member must be raised to the power indicated, and the proper sign placed between the results. In exercises like the 31st, the root indicated must be extracted from each member, and the two results connected by the proper sign.

Simplify:

35. $4+3ab>2x-(a-b)^2$.

36. $(x-1)^3<x^3-x^2+4x$.

37. $x^3-y^3<(x-y)(x^2+y^2)$.

Solve:

38. $x+4>2-3x$.

39. $(x+1)^2<x^2+3x-5$.

40. $4x-3>\frac{3x}{2}-\frac{3}{5}$.

41. $3x-2<\frac{x}{2}+7\frac{1}{2}$.

42. $2\sqrt{x^2}>\sqrt{4(x-1)^2+12x^2-7}$.

43. $\sqrt{m^2+5mx+4x^2}<2x+3m$.

44. $\sqrt{2}-3\sqrt{x}<\sqrt{4+9x-12}$.

45. $\frac{4x-2}{3}>\frac{3-5x}{7}$.

46. $\sqrt{\frac{2x^2-3}{2}}>x+5$.

47. $2x-3y>4$,
$4x+y>12$.

48. $3x-1>x+3y$,
$x(1-3x)>4x-3x^2-2y$.

49. $2x(2x-3)>(2x+1)^2-3y$,
$4x-y>x+4$.

50. $\sqrt{x^2-3x+y}>x+2$,
$x-y>3$.

51. $\sqrt{(2x-1)(y+x)}>\sqrt{2(xy+x^2)+4}$,
$5x+2y>8$.

52. Prove that $a^2+b^2>2ab$.

53. Prove that a fraction increased by its reciprocal value is greater than 2.

54. Let $\frac{a}{m}, \frac{b}{n}, \frac{c}{p}, \ldots\ldots, \frac{h}{u}$, be fractions whose denominators have the same sign, and arranged in the order of their magnitude, so that $\frac{a}{m}$ may be the smallest. Prove that

$$\frac{a+b+c+\cdots\cdots+h}{m+n+p+\cdots\cdots+u} > \frac{a}{m} \text{ and } < \frac{h}{u}.$$

An inequality of the second degree can always be reduced to the form $ax^2+bx+c>0$, or $ax^2+bx+c<0$.

To *solve* it is to find between what limits the value of x must lie to satisfy the inequality.

Solve:

55. $x^2+x>7x-10$.

56. $x^2+19>1+8x$.

57. $29-11x>-x(1+x)$.

58. $x^2+x-6>0$.

59. $6x^2+7x+2<0$.

60. $x^2+2x-15<0$.

61. $x^2-4x+7<0$.

62. $x^2-10x+32<0$.

63. $x^2+16x-80<0$.

64. $4x+21-x^2>0$.

65. $4x+12-x^2>0$.

66. $15-x-2x^2>0$.

67. $17x-6x^2-5<0$.

68. $38x-7-15x^2<0$.

69. $9x<20x^2+1$.

Ex. 35.

Exponents, Multiplication.

$a^m \times a^n = a^{m+n}$.

$(-a)^n = \begin{cases} -a^n \text{ when } n \text{ is odd.} \\ +a^n \text{ when } n \text{ is even.} \end{cases}$

$a^m \times b^m = (ab)^m$.

If p is even:

$(a-b)^p = (b-a)^p, \quad (a-b)^4 = (b-a)^4$.

$(a-b)^{p+1} = -(b-a)^{p+1}, (a-b)^3 = -(b-a)^3$.

$(a^m)^n = a^{mn} = a^{nm} = (a^n)^m$.

Simplify :

1. $-(+8)^2+(-2)^3$.
2. $(-7)^3-(-3)^7$.
3. $b^{m-3}\times b^2$.
4. $d^{a-5}\times d^6$.
5. $a^{3x-y}a^{2x-y}$.
6. $p^{3a-b}p^{7b-2a}$.
7. $(a+b)^n\times(a+b)^m$.
8. $(x-y)^{m-1}(x-y)$.
9. $(m^x-n^y)^2$.
10. $(p^{m-1}+q^{n-2})^2$.
11. $(a^m+b^n)^3$.
12. $(m^a+n^b+p^c)^2$.
13. $(-y)^{2n+1}\times(-y)^3$.
14. $(-d)^{2n-1}\times(-d)^5$.
15. $(-c)^{2m+1}\times(-c)^{2n-1}$.
16. $(-a)^{2m}\times(-a)^7$.
17. $(-m)^7\times(-m)^{2p}$.
18. $(-x)^{2m+1}\times(-x)^5$.
19. $(-xy)^{2n-1}\times(-xy)^3$.
20. $\left(-\frac{x}{y}\right)^7\times\left(-\frac{x}{y}\right)^8$.
21. $\left(-\frac{a}{b}\right)^3\times\left(\frac{a}{b}\right)^7$.
22. $\left(-\frac{m}{n}\right)^9\times\left(-\frac{n}{m}\right)^{10}$.
23. $\left(-\frac{b}{x}\right)^{12}\times\left(-\frac{x}{c}\right)^9$.
24. $\left(-\frac{x}{y}\right)^{2m}\times\left(-\frac{y}{x}\right)^{2m-1}$.
25. $\left(-\frac{m}{n}\right)^{2m-1}\times\left(-\frac{n}{m}\right)^{2m+1}$.
26. $\left(-\frac{ab}{xy}\right)^8\times\left(-\frac{ax}{by}\right)^{11}$.
27. $\left(\frac{7-c}{m-5}\right)^4\times\left(\frac{5-m}{7-c}\right)^6$.
28. $\left(\frac{m-p}{x-y}\right)^8\times\left(\frac{y-x}{p-m}\right)^{10}$.
29. $\left(\frac{8-a}{x-y}\right)^4\times\left(\frac{x-y}{a-8}\right)^3$.
30. $\left(\frac{x-y}{a-b}\right)^{2m}\times\left(\frac{b-a}{x-y}\right)^{2m+1}$.
31. $\left(\frac{a-b}{x-y}\right)^{2n}\times\left(\frac{y-x}{b-a}\right)^{2n-1}$.
32. $\left(\frac{a^2-b^2}{x^4-y^4}\right)^5\times\left(\frac{x^4-y^4}{b^2-a^2}\right)^7$.
33. $\left(\frac{a^2-b^2}{x^3-y^3}\right)^3\times\left(\frac{x^6-y^6}{b^2-a^2}\right)^4$.
34. $\left(\frac{m^8-n^8}{x^5-y^5}\right)^{2m}\times\left(\frac{y^{10}-x^{10}}{n^4-m^4}\right)^m\times\left(\frac{x^5-y^5}{m^4+n^4}\right)^m$.

35. $(a^4)^3$.

36. $[(b^3)^2]^5$.

37. $(x^{m+1})^2$.

38. $(-a^3)^2$.

39. $(-a^2)^3$.

40. $(-m^3)^7$.

41. $(-y^3)^{2m}$.

42. $(-y^{2m})^3$.

43. $(-y^{3m})^2$.

44. $(-b^2)^{2m-1}$.

45. $(-b^{2n})^{2n-1}$.

46. $(m^n p)^5$.

47. $(5a^{n-1}b^3)^2$.

48. $(3a^3b^{n-2})^3$.

49. $(7a^2x^{n-2}y^{m+1})^3$.

50. $(8a^nb^{n-1}c^{n-2})^3$.

51. $\left(\frac{a^2b^3}{c^4d^5}\right)^2$.

52. $\frac{(a^2x^3)^4}{(ax)^5}$.

53. $\frac{(ab)^6}{(a^2b^3)^3}$.

54. $\frac{(a^6b^3)^2}{(a^5b^2)^4}$.

55. $\left(\frac{a^9b^{28}c^{47}}{d^{10}e^{29}}\right)^{17} \times \left(\frac{d^9e^{26}}{a^8b^{25}c^{42}}\right)^{19}$.

56. $\frac{[(8x-6y)^{2a}]^{5a}}{[(4x-3y)^{5a}]^{2a}}$.

57. $\left[\left(\frac{m^5n^3}{p^2q^2}\right)^3 \times \left(\frac{mq^3}{n}\right)^4\right] \div \left[\left(\frac{m^3n}{p^2q^3}\right)^5 \div \left(\frac{m^2q^3}{n^2p}\right)^6\right]$.

58. $\left[\left(\frac{a^4b^5}{c^2x^3}\right)^5 \div \left(\frac{ax^3}{bc^4}\right)^4\right] \div \left[\left(\frac{ab^3}{c^2x^3}\right)^6 \div \left(\frac{a^2x^5}{b^2c^3}\right)^5\right]$.

Ex. 36.

Exponents, Division.

$$a^m \div a^n = \frac{a^m}{a^n} = \begin{cases} a^{m-n}, \text{ when } m > n. \\ a^0 = 1, \text{ when } m = n. \\ a^{-(n-m)} = \frac{1}{a^{n-m}}, \text{ when } m < n. \end{cases}$$

$a^5 \div a^3 = a^2$. $\quad a^5 \div a^5 = a^0 = 1$.

$a^5 \div a^9 = a^{-4} = \frac{1}{a^4}$.

$a^m \div b^m = (a \div b)^m = \frac{a^m}{b^m} = \left(\frac{a}{b}\right)^m$.

$1 \div b^n = (1 \div b)^n = \frac{1}{b^n} = \left(\frac{1}{b}\right)^n$.

$a^m \div a^m = a^0 = 1$.

$a^{3m-2n} = \frac{a^{3m}}{a^{2n}}$.

Simplify:

1. $c^a \div c$.
2. $m^{a+3} \div m^3$.
3. $m^{y+5} \div m^6$.
4. $m^{x+y} \div m^y$.
5. $x^a \div x^{3b}$.
6. $x^{a+b} \div x^{a-b}$.
7. $x^{a-b} \div x^{b-a}$.
8. $a^{3x+2y} \div a^{2x-3y}$.
9. $(x+y)^{a-b} \div (x+y)^{b-a}$.
10. $\dfrac{a^{p+3q}\, b^{7p-8q}}{a^{4p-7q}\, b^{3p-11q}} \div \dfrac{a^{2(p-3q)}\, b^{5p+6q}}{a^{6p-17q}\, b^{2(p+2q)}}$.
11. $(x^{4p} - 1) \div (x^p - 1)$.
12. $(9x^p + 3x^{4p} + 14x^{3p} + 2) \div (1 + 5x^p + x^{2p})$.
13. $(a^n - b^n) \div (a - b)$.
14. $(a^{2n} - b^{2n}) \div (a + b)$.
15. $\dfrac{(a^{n+x} - a^n)(a^n - a^{n-x})}{(a^{n+x} - a^n) - (a^n - a^{n-x})}$.
16. $\left(\dfrac{5a^3}{3c}\right)^n \div \left(\dfrac{15a^2}{6c^3}\right)^n$.
17. $\dfrac{(20a^4 + 8a^2b^2 - 12b^4)^x}{[4(a^2 + b^2)]^x}$.
18. $\dfrac{(64m^2 - 49n^2)^b}{(8m - 7n)^b}$.
19. $\dfrac{1}{0.25^3}$.
20. $\dfrac{1}{(0.5)^6}$.
21. $\dfrac{1}{(\frac{1}{8})^5}$.

Reduce to the simplest form:

22. $\left(\dfrac{3mn}{5pq}\right)^4 \times \left(\dfrac{5p}{6m}\right)^3 \times \left(\dfrac{4n}{3q}\right)^2$.
23. $\left(\dfrac{m+n}{p-q}\right)^4 \times \left(\dfrac{3}{m+n}\right)^2 \times \left(\dfrac{p-q}{3(m+n)}\right)^5$.

Write the following expressions without using the exponent zero or negative exponents, and simplify the results:

24. x^0.
25. a^{-3}.
26. $\dfrac{1}{x^0}$.
27. $\dfrac{1}{a^{-4}}$.
28. x^{-0}.
29. $\dfrac{1}{x^{-0}}$.
30. $a^3 \times a^0$.
31. $a^3 \times a^{-0}$.
32. $\dfrac{a^5}{q^{-0}}$.

33. $m^{-3} \times x^{-0}$.

34. $m^{-3} \times x^{0}$.

35. $\dfrac{m^{-3}}{x^{-0}}$.

36. $b^{m} \times b^{0}$.

37. $b^{m} d^{-0}$.

38. $b^{m} \times d^{-5}$.

39. $x^{0} \times y^{0}$.

40. $x^{-0} \times y^{-c}$.

41. $(x+y)^{0}$.

42. $x^{0} + y^{0}$.

43. $x^{0} + x^{0}$.

44. $x^{0} x^{0}$.

45. $0^{x} \times 0^{x}$.

46. $x^{-0} + y^{-0}$.

47. $(x+y)^{-0}$.

48. 0^{-x}.

49. 0^{-4}.

50. $(-a)^{-2}$.

51. $(-x)^{-4}$.

52. $(-x)^{-3}$.

53. $(-x^{3})^{-5}$.

54. $\dfrac{a^{2}}{a^{-3}}$.

55. $\dfrac{1}{m^{-2}}$.

56. $(-x^{2})^{0}$.

57. $\dfrac{1}{m^{0}}$.

58. $\dfrac{1}{m^{-0}}$.

59. $\dfrac{a^{0}}{b^{m}}$.

60. $\dfrac{a^{0}}{b^{-m}}$.

61. $\dfrac{x^{-n}}{y^{0}}$.

62. $\dfrac{x^{-n}}{y^{-0}}$.

63. $\dfrac{m^{-5}}{n^{-3}}$.

64. $\dfrac{a^{-x}}{b^{-x}}$.

65. $\dfrac{a^{-m}}{b^{-n}}$.

66. $\dfrac{m^{-6}}{m^{-5}}$.

67. $\dfrac{a^{n-4}}{a^{-5}}$.

68. $\dfrac{p^{-3}}{p^{m-4}}$.

69. $\dfrac{a^{m-2}}{b^{n-3}}$.

70. $\left(\dfrac{1}{n}\right)^{-4}$.

71. $\left(\dfrac{x}{y}\right)^{-1}$.

72. $\left(\dfrac{x}{y}\right)^{0}$.

73. $\left(\dfrac{y}{z}\right)^{-0}$.

74. $\dfrac{1}{m}\left(\dfrac{a}{m}\right)^{-1}$.

75. $p^{2}\left(\dfrac{p}{q}\right)^{-2}$.

76. $m(1-m)^{-1}$.

77. $n(1-n)^{-0}$.

78. $(1+x)(1-x^{2})^{-1}$.

79. $(1+y)^{2}(1-y^{2})^{-2}$.

80. $(m-n)^{3}(m-n)^{-4}$.

81. $(a-1)^{-6} \times (1-a)^{3}$.

82. $\dfrac{3a^{0}b^{-2}c^{-4}}{5a^{-1}b^{-3}c^{-5}}$.

83. $\dfrac{4(a^{0}+b^{0})^{-1} \times x^{-5}}{5x^{-3}y^{-4}}$.

84. $\dfrac{a^0b^{-2}}{9(x^0+y^0+z^0)^{-2}c^3}$.

85. $\dfrac{a^2b^{-4}}{x^{-3}y^{-5}} \times \dfrac{a^{-2}b^4}{x^5y^3}$.

86. $\dfrac{8a^{-5}}{a^2} \times \dfrac{(a^0+y^0)^{-3}}{a^{-10}}$.

87. $\dfrac{m^{2x+4}m^{-(x+5)}}{n^{x-3}}$.

88. $(a^{2x-3})^{-2}$.

89. $(m^{3x-5}-n^{5x-3})^{-x}$.

90. $\left(\dfrac{a^{-3x}+b^{-2y}}{a^{-6x}-b^{-4y}}\right)^{-3}$.

91. $(x^{3-m}+y^{4-n})^{-2}$.

92. $\left(\dfrac{a^{-1}+b^{-1}}{a^{-2}-b^{-2}}\right)^{-2} \div \left(\dfrac{a^{-2}+b^{-2}}{a^{-4}-b^{-4}}\right)^{-3}$.

93. $\left(\dfrac{m^{-2}+n^{-2}}{m^{-4}-n^{-4}}\right)^{-3} \div \left(\dfrac{m^{-1}+n^{-1}}{n^{-2}-m^{-2}}\right)^{-2}$.

Free the following expressions from the form of fractions by using negative exponents:

94. $\dfrac{3}{m}$.

95. $\dfrac{x}{(1-y)^3}$.

96. $\dfrac{x}{(-z^3)}$.

97. $\dfrac{b}{(-p)^4}$.

98. $\dfrac{b}{(-p^4)}$.

99. $a^5 \times \dfrac{1}{a^3}$.

100. $\left(\dfrac{a^m}{xy^2}\right)^3$.

101. $\dfrac{b^n}{by^5}$.

102. $\dfrac{c^4}{c^mz^p}$.

103. $\dfrac{x^m}{x^5}+\dfrac{y^n}{y^3}$.

104. $\dfrac{1}{\left(\dfrac{1}{x^2}+\dfrac{1}{y}\right)^2} \times \dfrac{1}{\left(\dfrac{1}{x^3}-\dfrac{1}{y^4}\right)^3}$.

105. $\dfrac{1}{(x+y)^2} \div \dfrac{1}{x^2+y^2}$.

106. $\dfrac{1}{m^2} \times \dfrac{1}{n^2} \div \dfrac{3}{m^4n^4}$.

Calculate the numerical values of the following expressions:

107. $\dfrac{1}{2^0}$.

108. $\dfrac{1}{2^{-1}}$.

109. $\dfrac{2}{3^{-2}}$.

110. $\dfrac{5}{(-4)^3}$.

111. $\dfrac{3}{(-5)^{-3}}$.

112. $\dfrac{2^{-4}}{4^{-2}}$.

113. 2^{-1}.

114. 1^{-1}.

115. 5^{-2}.

116. 100×5^{-3}.

117. $7^0(x+y)^0$.

118. 1^{-n}.

119. 25×5^{-2}.

120. 64×4^{-3}.

121. $(0.5)^{-2} \times 1000$.

122. $(0.3)^{-4} \times 810$.

123. $(-0.8)^{-2} \times 256$.

124. $\dfrac{10}{2^{-3}}$.

125. $\dfrac{5}{3^{-4}}$.

126. $\dfrac{2^x}{(-1)^{-3}}$.

127. $\dfrac{2^0}{(-1)^{-4}}$.

128. $(\frac{1}{5})^{-1}$.

129. $(\frac{1}{4})^{-2}$.

130. $2^{-1} - 2^{-2}$.

131. $9^{-2} - 3^{-4}$.

132. $\dfrac{1}{15^{-2}} - \dfrac{1}{4^{-4}}$.

133. $\dfrac{1}{5^{-4}} - \dfrac{1}{10^{-2}}$.

134. $\dfrac{\dfrac{1}{4^{-3}} - \dfrac{2}{10^{-2}}}{\dfrac{1}{2^{-2}} + \dfrac{1}{4^{-1}}}$.

135. $\left(\dfrac{4^{-5} - \dfrac{1}{3^{-2}}}{\dfrac{1}{5^{-4}} + 8^{-2}}\right)^0$.

136. $64 \times 2^{-4} + 32 \times 2^{-3} + 16 \times 2^{-2} + 8 \times 2^{-1} + 4 \times 2^0$.

Ex. 37.

Negative Exponents, Multiplication.

$a^m \times a^{-n} = a^{m-n}$. $\quad a^{-m} \times a^{-n} = a^{-(m+n)}$.

$(-a)^{-n} = \dfrac{1}{(-a)^n} = \pm\dfrac{1}{a^n}$.

$a^{-m} \times b^{-m} = (ab)^{-m}$. $\quad (a-b)^{-2m} = (b-a)^{-2m}$.

$(a-b)^{-(2m+1)} = -(b-a)^{-(2m+1)}$.

Perform the following operations:

1. $a^4 \times a^{-3}$.
2. $m^7 \times m^{-3}$.
3. $x^{10} \times x^{-6}$.
4. $z^{-6} \times z^4$.
5. $a^5 a^{-6}$.
6. $h^m h^{-n}$.
7. $ab^{-5} \times a^{-1}b^4$.
8. $m^{-2}n^3 \times m^4 n^{-5}$.

9. $a^0b^{-m} \times a^3b^m$.

10. $a^m \times a^{-n} \times a^{-2m}$.

11. $a^{-4} \times a^{-3}$.

12. $b^{-2} \times b^{-5}$.

13. $x^{-m} \times x^{-p}$.

14. $y^{-a} \times y^{-c}$.

15. $x^{-2m} \times x^{-3m}$.

16. $y^{-5x} \times y^{-x}$.

17. $a^{-1} \times a^{-2b}$.

18. $a^{-2x}a^{-2y}$.

19. $(-a)^{-2}$.

20. $(-b)^{-3}$.

21. $(-x)^{-7}$.

22. $(-y)^{-10}$.

23. $(-m)^{-2} \times (-m)^{-3}$.

24. $(-x)^{-1} \times (-x)^{-3}$.

25. $x^{-5}(-x)^{-4}$.

26. $x^{-4}(-x)^{-5}$.

27. $(-y)^{-m} \times (-y)^{-m}$.

28. $(a^{-4} + a^{-3} + a^{-2} + a^{-1}) \times a^{-3}$.

29. $(y^{-5} + y^{-4} + y^{-3} + y^{-2} + y^{-1} + 1) \times (-y)^{-5}$.

30. $\dfrac{a^{-1}}{b^{-2}} \times \dfrac{a^{-2}}{b^{-1}}$.

31. $\left(\dfrac{1}{x}\right)^{-1} \times \left(\dfrac{y}{x}\right)^{-2}$.

32. $\dfrac{a^{-1}b^{-2}}{m^{-3}n^{-4}} \times \dfrac{a^{-2}b^{-3}}{m^{-2}n^{-1}} \times \dfrac{m^{-6}n^{-7}}{a^{-3}b^{-5}}$.

33. $(x^{-1} - 2x^{-2} + 3x^{-3}) \times (x^0 + x^{-1} - x^{-2})$.

34. $4^{-2} \times 3^{-2}$.

35. $5^{-1} \times 6^{-1}$.

36. $8^{-2} \times 5^{-2}$.

37. $1^{-3} \times 2^{-3} \times 3^{-3}$.

38. $5^{-4} \times 2^{-4}$.

39. $(4-1)^{-2}$.

40. $(1-4)^{-2}$.

41. $(5-2)^{-3}$.

42. $-(2-5)^{-3}$.

43. $(\frac{2}{3})^{-2} \times (\frac{3}{4})^{-2}$.

44. $0.2^{-4} \times 0.5^{-4}$.

45. $(\frac{1}{2}a)^{-1} \times (2c)^{-1}$.

46. $(\frac{3}{5}x)^{-4} \times (\frac{5}{3}y)^{-4}$.

47. $(a-b)^{-4}(b-a)^{-4}$.

48. $(x^2 - y^2)^{-6}(y^2 - x^2)^{-8}$.

49. $(3m^5 - 2n)^{-2}(2n - 3m^5)^{-2}$.

50. $(1-x^2)^{-7}(x^2-1)^{-7}$.

51. $\left(\frac{a^2}{b}+\frac{m^{-1}}{n^{-2}}\right)^{-3}\left(\frac{a^2}{b}-\frac{m^{-1}}{n^{-2}}\right)^{-3}$.

52. $\left(\frac{x}{y^3}-\frac{z^{-1}}{y^{-4}}\right)^{-7}\left(\frac{x}{y^3}+\frac{z^{-1}}{y^{-4}}\right)^{-7}$.

53. $\left(\frac{3x}{a}+\frac{2m^{-1}}{b^{-2}}\right)^{-2}\left(\frac{2m^{-1}}{b^{-2}}-\frac{3x}{a}\right)^{-2}$.

54. $\left(\frac{x^{-3}}{y^{-4}}+\frac{a^{-1}}{z^2}\right)^{-3}\left(\frac{a^{-1}}{z^2}-\frac{x^{-3}}{y^{-4}}\right)^{-3}$.

55. $\left(\frac{2}{y^{-3}}+\frac{1}{z^{-1}}\right)^{-6}\left(\frac{1}{z^{-1}}-\frac{2}{y^{-3}}\right)^{-6}$.

Ex. 38.

Negative Exponents, Powers, Division.

$(a^{-m})^n = a^{-mn} = (a^{-n})^m$.

$(a^m)^{-n} = a^{-mn} = (a^{-n})^m = (a^n)^{-m}$.

$(a^{-m})^{-n} = a^{mn}$.

$a^{-m} \div a^{-n} = a^{-m+n}$. $\quad a^{-5} \div a^{-3} = a^{-2} = \frac{1}{a^2}$.

$a^{-m} \div a^n = a^{-(m+n)}$. $\quad a^{-5} \div a^{-5} = a^0 = 1$.

$a^m \div a^{-n} = a^{m+n}$. $\quad a^{-5} \div a^{-8} = a^3$.

$a^{-m} \div b^{-m} = \left(\frac{a}{b}\right)^{-m} = \left(\frac{b}{a}\right)^m$. $\quad 1 \div b^{-m} = \left(\frac{1}{b}\right)^{-m} = b^m$.

Find the values of:

1. $(2^{-3})^2$.
2. $(2^{-2})^3$.
3. $(5^{-3})^{-4}$.
4. $(10^{-2})^4$.
5. $(3^{-2})^{-3}$.
6. $(5^{-3})^{-1}$.
7. $[(2^{-3})^{-2}]^2$.
8. $(100^{-1})^{-3}$.
9. $(\frac{2}{3})^{-3}$.
10. $(\frac{1}{5})^{-4}$.
11. $2^{-3} \times (\frac{3}{4})^{-2}$.
12. $(\frac{3}{5})^{-3} \div (\frac{9}{10})^{-2}$.
13. $[(2a)^{-2}]^{-3}$.
14. $[(4a)^{-2}]^2$.
15. $(x^{-3})^{-2}$.
16. $(y^2)^{-2}$.
17. $(-1^{-1})^{-2}$.
18. $(-a)^{-2}$.
19. $[(-b)^2]^{-2}$.
20. $(-m)^{-3}$.
21. $(a^2b^{-1})^{-2}$.

22. $(m^{-1}n^{-3})^{-1}$.
23. $(a^{-2}b^{-5})^{2}$.
24. $(-a^{-1}b^{-2})^{-3}$.
25. $(-x^{-2}y^{-4})^{-5}$.
26. $(-x^{-1}y^{-2})^{-2}$.
27. $(a^{-m})^{-n}$.
28. $(bn)^{-3}$.
29. $(d^{-5})^{-p}$.
30. $(c^{-x})^{2}$.
31. $[(-m)^{-2x}]^{y}$.
32. $[(-a)^{2y}]^{-1}$.
33. $(a^{-m}b^{n})^{-2}$.
34. $(b^{-1}c^{-y})^{-3}$.
35. $(-x^{-a}y^{-m})^{2n}$.
36. $(-3a^{-2})^{-2x}$.
37. $[(-a^{-1})^{-2}]^{-3}$.
38. $[(-a^{-2})^{-2n}]^{-(2n+1)}$.
39. $[(a^{-1})^{-3}]^{-2n}$.
41. $\left(\frac{a^{-1}b^{2}}{c^{-m}d^{-n}}\right)^{-2}$.
43. $\left(\frac{x^{-1}y}{az^{0}}\right)^{-1}$.
40. $[(-m^{-p})^{-2q}]^{-x}$.
42. $\left(\frac{a^{0}bc^{-3}}{x^{-2}y^{4}}\right)^{-m}$.
44. $\left(\frac{mn^{0}}{x^{-p}y^{-q}}\right)^{-1}$.
45. $\left(\frac{m+n}{m-n}\right)^{-3}\times\left(\frac{m-n}{m+n}\right)^{-2}$.
46. $\left(\frac{m+n}{m-n}\right)^{-2}\times\left(\frac{p+q}{m-n}\right)^{2}$.
47. $\left(\frac{x-y}{a-b}\right)^{2}\times\left(\frac{x-y}{a+b}\right)^{-2}$.
48. $\left(\frac{x-y}{p+q}\right)^{-4}\times\left(\frac{x+y}{p+q}\right)^{4}\times\left(\frac{x+y}{x-y}\right)^{-5}$.
49. $\left(\frac{m+n}{m-n}\right)^{-4}\times(m-n)^{-5}\times\left(\frac{1}{m+n}\right)^{-5}$.
50. $\left(\frac{1}{1+x}\right)^{5}\times\left(\frac{1}{1-x}\right)^{-7}\times\left(\frac{x-1}{x+1}\right)^{-6}$.
51. $\left[\left(\frac{a-b}{x+y}\right)^{-3}\right]^{2}\times\left[\left(\frac{x+y}{a-b}\right)^{-2}\right]^{3}$.
52. $\left[\left(\frac{a-b}{x+y}\right)^{-2}\right]^{-5}\times\left[\left(\frac{x+y}{a-b}\right)^{-5}\right]^{-3}$.
53. $\left[\left(\frac{a+b}{c-d}\right)^{-2}\right]^{7}\times\left[\left(\frac{c-d}{a+b}\right)^{-6}\right]^{2}$.

54. $\left[\left(\frac{a+b}{x-y}\right)^{-4}\right]^{-5} \times \left[\left(\frac{a+b}{y-x}\right)^{-7}\right]^{3}$.

55. $\left[\left(\frac{c-d}{m-n}\right)^{-3}\right]^{-5} \times \left[\left(\frac{d-c}{n-m}\right)^{-7}\right]^{2}$.

56. $2^{-4} \div 2^{-5}$.

57. $3^{-4} \div 3^{-7}$.

58. $5^{-1} \div 5^{-3}$.

59. $7^{-4} \div 7^{-2}$.

60. $6^{-7} \div 6^{-3}$.

61. $10^{-8} \div 10^{-6}$.

62. $7^{-9} \div 7^{-5}$.

63. $3^{2} \div 3^{-2}$.

64. $3^{-2} \div 3^{2}$.

65. $7^{-3} \div 7^{3}$.

66. $25^{-8} \div 25^{-8}$.

67. $(\frac{3}{4})^{-5} \div (\frac{3}{4})^{-5}$.

68. $(\frac{2}{3})^{-4} \div (\frac{2}{3})^{-7}$.

69. $(\frac{1}{2})^{2} \div (\frac{1}{2})^{-3}$.

70. $(\frac{2}{5})^{-5} \div (\frac{2}{5})^{-2}$.

71. $(0.3)^{-1} \div (0.3)^{-3}$.

72. $(0.7)^{-6} \div (0.7)^{-6}$.

73. $(0.4)^{-7} \div (0.4)^{-4}$.

74. $(0.1)^{-5} \div (0.1)^{-2}$.

75. $(0.2)^{-5} \div (0.2)^{-2}$.

Perform the following operations, and give the results freed from negative exponents:

76. $a^{-3} \div a^{-5}$.

77. $a^{-4} \div a^{-3}$.

78. $a^{-7} \div a^{-5}$.

79. $m^{-x} \div m^{-2x}$.

80. $m^{-(x+2)} \div m^{-2x}$.

81. $x^{-2a} \div x^{-3a}$.

82. $y^{-a} \div y^{-m}$.

83. $b^{-m+1} \div b^{-(m+1)}$.

84. $c^{-(2m+3x)} \div c^{-(2m-3x)}$.

85. $(a+b)^{-(x+y)} \div (a+b)^{-(x-y)}$.

86. $\frac{a^{-5}b^{-x}}{m^{-3}n^{-4}} \div \frac{a^{-6}b^{-(x+2)}}{m^{-7}n^{-3}}$.

87. $(a^{-6} - b^{-4}) \div (a^{-3} - b^{-2})$.

88. $(x^{-m} - 1) \div (x - 1)$.

89. $(x^{m} - 1) \div (x^{-1} - 1)$.

Find the values of:

90. $5^{-3} \div 10^{-3}$.

91. $3^{-5} \div 2^{-5}$.

92. $14^{-3} \div 7^{-3}$.

93. $8^{-5} \div 16^{-5}$.

94. $9^{-4} \div 3^{-4}$.

95. $1^{-7} \div 2^{-7}$.

96. $3^{-4} \div 1^{-4}$.

97. $(0.2)^{-3} \div (0.5)^{-3}$.

98. $(0.6)^{-2} \div (0.7)^{-2}$.

99. $3^{-5} \div (0.3)^{-5}$.

100. $(\frac{2}{5})^{-3} \div (\frac{4}{15})^{-3}$.

101. $(\frac{3}{4})^{-4} \div (\frac{9}{8})^{-4}$.

102. $(\frac{3}{7})^{-2} \div (\frac{9}{28})^{-2}$.

103. $\left(\frac{3\frac{2}{5}}{4\frac{1}{5}}\right)^{-3} \div \left(\frac{8\frac{1}{2}}{10\frac{1}{2}}\right)^{-3}$.

104. $\left(\frac{13\frac{1}{2}}{11\frac{1}{2}}\right)^{-5} \div \left(\frac{16\frac{1}{5}}{18\frac{2}{5}}\right)^{-5}$.

105. $(1 \div a^{-m})^2 \div \left(\frac{1}{a}\right)^{-2m}$.

106. $\left(\frac{a^2}{b^{-3}}\right)^{-2} \div \left(\frac{a^{-3}}{b^4}\right)^{-2}$.

107. $\left[x^{-6} - \left(\frac{1}{y^{-1}}\right)^{-2}\right] \div \left[x^{-3} + \left(\frac{1}{y^{-1}}\right)^{-1}\right]$

Ex. 39.

Fractional Exponents.

$$\sqrt[p]{a^{mp}} = a^m = a^{\frac{mp}{p}}.$$

$$\sqrt[m]{a^{-2}} = \sqrt[m]{\frac{1}{a^2}} = \frac{1}{a^{\frac{2}{m}}} = \frac{1}{\sqrt[m]{a^2}}.$$

$$\sqrt[m]{\frac{a^x}{b^y}} = \frac{\sqrt[m]{a^x}}{\sqrt[m]{b^y}} = \frac{a^{\frac{x}{m}}}{b^{\frac{y}{m}}}.$$

Express with fractional exponents:

1. $\sqrt{a^3}$.
2. $\sqrt{a^5}$.
3. $(\sqrt{a})^6$.
4. $\sqrt[3]{n^5}$.
5. $(\sqrt[4]{a})^5$.
6. $\sqrt{a^3b}$.
7. $\sqrt[3]{a^3b^2}$.
8. $\sqrt[3]{ab^5}$.
9. $\sqrt[3]{a^3 - 3a^2b + 3ab^2 - b^3}$.
10. $\sqrt{m^2 - 2mp + p^2}$.
11. $\sqrt[3]{27a^{-2}b^4}$.
12. $\sqrt[3]{\frac{m^5}{n^4}}$.
13. $\sqrt[4]{\frac{a^6}{b^8}}$.
14. $\sqrt[3]{x^4}$.
15. $\sqrt{m^3}$.
16. $\sqrt[3]{\sqrt{a^5}}$.
17. $\sqrt[4]{\sqrt[3]{a^7}}$.

Write the following expressions, using only integral and positive exponents:

18. $a^{\frac{1}{2}}$.
19. $1^{\frac{1}{2}}$.
20. $m^{\frac{1}{2}}$.
21. $x^{\frac{1}{3}}$.
22. $n^{\frac{1}{3}}$.
23. $b^{\frac{2}{3}}$.
24. $c^{\frac{1}{4}}$.
25. $h^{\frac{1}{5}}$.
26. $x^{\frac{a}{b}}$.
27. $y^{\frac{1}{m}}$.
28. $a^{-\frac{1}{2}}$.
29. $c^{-\frac{2}{3}}$.
30. $e^{-\frac{m}{n}}$.
31. $x^{-\frac{2m}{3}}$.
32. $y^{-\frac{a+b}{m+1}}$.
33. $z^{p-\frac{1}{2}}$.
34. $(x^3+y^3)^{\frac{1}{3}}$.
35. $x^{\frac{1}{2}}y^{\frac{1}{3}}$.
36. $(m^3-n^3)^{-\frac{1}{2}}$.
37. $\left(\frac{a^{-1}}{b^{-2}}\right)^{-\frac{1}{2}}$.
38. $\left(\frac{m^5}{n^4}\right)^{\frac{2}{3}}$.
39. $\left(\frac{x^{-3}}{y^{-4}}\right)^{-\frac{1}{2}}$.
40. $\left(\frac{a^2}{b^{-2}}\right)^{-1\frac{1}{2}}$.

Find the numerical values of the following:

41. $4^{\frac{1}{2}}$.
42. $27^{\frac{1}{3}}$.
43. $625^{\frac{1}{4}}$.
44. $16^{\frac{1}{4}}$.
45. $9^{\frac{1}{2}}$.
46. $64^{\frac{1}{2}}$.
47. $3125^{\frac{1}{5}}$.
48. $216^{\frac{1}{3}}$.
49. $49^{\frac{1}{2}}$.
50. $729^{\frac{1}{3}}$.
51. $16^{\frac{1}{2}}$.
52. $81^{\frac{1}{2}}$.
53. $125^{\frac{1}{3}}$.
54. $32^{\frac{2}{5}}$.
55. $81^{\frac{1}{4}}$.
56. $256^{\frac{1}{4}}$.
57. $512^{\frac{1}{3}}$.
58. $128^{\frac{1}{7}}$.
59. $16^{-\frac{1}{4}}$.
60. $64^{-\frac{1}{3}}$.
61. $1024^{-\frac{1}{2}}$.
62. $64^{-\frac{1}{6}}$.
63. $512^{-\frac{1}{9}}$.
64. $81^{-\frac{1}{4}}$.
65. $625^{-\frac{1}{2}}$.
66. $(\frac{25}{36})^{\frac{1}{2}}$.
67. $(\frac{8}{27})^{\frac{1}{3}}$.
68. $(\frac{625}{1296})^{\frac{1}{4}}$.
69. $(\frac{32}{243})^{\frac{1}{5}}$.
70. $(\frac{1}{64})^{\frac{1}{6}}$.
71. $(\frac{25}{36})^{-\frac{1}{2}}$.
72. $(\frac{8}{27})^{-\frac{1}{3}}$.
73. $(\frac{625}{1296})^{-\frac{1}{4}}$.
74. $(\frac{32}{243})^{-\frac{1}{5}}$.
75. $(\frac{1}{64})^{-\frac{1}{6}}$.
76. $(3\frac{3}{8})^{-\frac{1}{3}}$.
77. $(6\frac{1}{4})^{\frac{1}{2}}$.
78. $(7\frac{19}{32})^{\frac{2}{5}}$.
79. $(7\frac{19}{32})^{-\frac{2}{5}}$.
80. $(0.25)^{\frac{1}{2}}$.
81. $(0.027)^{\frac{1}{3}}$.
82. $(0.125)^{-\frac{1}{3}}$.
83. $(-0.027)^{\frac{1}{3}}$.
84. $(-0.00032)^{\frac{2}{5}}$.

Ex. 40.

Fractional and Negative Exponents.

$a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}} = a^{\frac{2}{2}}$.

$a^{-\frac{m}{n}} \times a^{\frac{p}{n}} = a^{\frac{p-m}{n}}$.

$\left(a^{\frac{m}{n}}\right)^{\frac{x}{b}} = a^{\frac{mx}{bn}}$.

$a^{-\frac{m}{n}} + a^{-\frac{x}{y}} = a^{\frac{x}{y}-\frac{m}{n}}$ $\quad a^{-\frac{m}{n}} + a^{\frac{x}{y}} = a^{-\left(\frac{m}{n}+\frac{x}{y}\right)}$.

$a^{\frac{m}{n}} + a^{-\frac{x}{y}} = a^{\frac{m}{n}+\frac{x}{y}}$.

Add:

1. $a^{\frac{1}{2}}$; $3a^{\frac{1}{2}}$.
2. $\frac{1}{2}b^{\frac{1}{2}}$; $4b^{-\frac{1}{2}}$.
3. $5m^{\frac{1}{2}}$; $-10m^{-\frac{1}{2}}$.
4. $5x^{-\frac{1}{3}}$; $4x^{-\frac{1}{3}}$.
5. $0.5m^{-\frac{1}{4}}$; $0.31m^{-\frac{1}{4}}$.
6. $4c^{-\frac{1}{2}}$; $-0.6c^{-\frac{1}{2}}$; $-3c^{-\frac{1}{2}}$.
7. d^{-4}; $3d^{-4}$; $-5.3d^{-4}$.
8. $x^{-2} - x^{-3}y^{\frac{1}{2}} - 3x^{-4}y^{\frac{1}{2}}$; $2x^{-3}y^{\frac{1}{2}} - 2x^{-2} + 5x^{-4}y^{\frac{1}{2}}$; $-x^{-4}y^{\frac{1}{2}} + x^{-2} - 3x^{-3}y^{\frac{1}{2}}$.
9. $x^{-\frac{1}{2}} - 2ax^{-\frac{1}{2}} + 2bx^{-\frac{1}{2}}$; $3ax^{-\frac{1}{2}} - mx^{-\frac{1}{2}} - cx^{-\frac{1}{2}}$; $bx^{-\frac{1}{2}} + dx^{-\frac{1}{2}} - 2x^{-\frac{1}{2}}$.
10. $y^{-\frac{1}{2}} + by^{-\frac{1}{2}} - cy^{\frac{1}{2}}$; $4y^{-\frac{1}{2}} - cy^{-\frac{1}{2}} + 2y^{\frac{1}{2}}$; $my^{-\frac{1}{2}} - 2y^{-\frac{1}{2}}$; $4by^{-\frac{1}{2}} - my^{-\frac{1}{2}}$; $-y^{-\frac{1}{2}} + y^{\frac{1}{2}}$.

From the first expression subtract the following expressions in succession:

11. $b^{-\frac{1}{2}}$; $2b^{-\frac{1}{2}}$; $ab^{-\frac{1}{2}}$.
12. $d^{-\frac{1}{2}}$; $\frac{1}{4}d^{-\frac{1}{2}}$; $0.7d^{-\frac{1}{2}}$.
13. e^{-x}; $-2e^{-x}$; $2.5e^{-x}$.
14. $0.8h^{-4}$; $-0.05h^{-4}$; $2h^{-4}$.
15. $-2.3m^{-5}$; $0.7m^{-5}$; $-7m^{-5}$.
16. $\frac{1}{2}p^{-3}$; $0.7p^{-3}$; $-1\frac{1}{2}p^{-3}$.
17. $3x^{-1} - 2x^{-2} + 0.5x^{-3}$; $4x^{-2} + 2x^{-1} - \frac{3}{5}x^{-3}$; $0.7x^{-2} - 0.4x^{-1} + 2x^{-3}$; $-x^{-3} - x^{-1} + x^{-2}$.

Perform the following multiplications:

18. $a^{\frac{1}{2}} \times a^{\frac{3}{2}}$.
19. $b^{-\frac{3}{2}} \times b^{\frac{3}{2}}$.
20. $a^{-\frac{1}{2}} \times a^{-\frac{1}{2}}$.
21. $c^{-\frac{1}{2}} \times c^{-\frac{1}{2}}$.
22. $d^{-\frac{1}{2}} \times d^{-1}$.
23. $m^{-\frac{3}{2}} \times m^{-\frac{1}{2}}$.
24. $m^{-\frac{1}{2}} \times m^{\frac{1}{2}}$.
25. $x^{-\frac{1}{m}} \times x^{\frac{1}{m}}$.
26. $x^{\frac{m}{n}} \times x^{\frac{p}{q}}$.
27. $y^{-\frac{a}{c}} \times y^{-\frac{m}{n}}$.
28. $1^{-\frac{1}{m}} \times a^{-\frac{x}{y}}$.
29. $z^{-\frac{c}{m}} \times z^{-0}$.
30. $u^{-1} \times u^{\frac{3}{2}}$.
31. $d^{-\frac{2}{m}} \times d^{-\frac{n}{2m}}$.
32. $e^{-2x} \times e^{-\frac{x}{y}}$.
33. $(a^{\frac{3}{2}})^{\frac{a}{2b}}$.
34. $(b^{\frac{1}{2}})^{\frac{1}{2}}$.
35. $(h^{\frac{1}{2}})^{\frac{1}{2}}$.
36. $(m^{\frac{1}{x}})^{\frac{a}{b}}$.
37. $(n^{-\frac{a}{b}})^{\frac{c}{d}}$.
38. $(x^{\frac{r}{2}} + 2x^{\frac{r}{4}}y^{p} + 3y^{2p}) \times (x^{\frac{r}{2}} - 2x^{\frac{r}{4}}y^{p} + 3y^{2p})$.
39. $(x^{\frac{p}{2}} - x^{\frac{p}{4}}y^{\frac{q}{4}} + y^{\frac{q}{2}}) \times (x^{\frac{p}{2}} + x^{\frac{p}{4}}y^{\frac{q}{4}} + y^{\frac{q}{2}})$.
40. $(x^{-1} + x^{-\frac{1}{2}}y^{-\frac{1}{2}} + y^{-1}) \times (x^{-1} - x^{-\frac{1}{2}}y^{-\frac{1}{2}} + y^{-1})$.
41. $(c^{-\frac{4}{m}} - 3c^{-\frac{3}{m}} + 5c^{-\frac{2}{m}} - c^{-\frac{1}{m}}) \times (c^{-\frac{3}{m}} - 2c^{-\frac{2}{m}} - c^{-\frac{1}{m}})$.
42. $(\frac{1}{2}d^{-\frac{1}{x}} + \frac{1}{3}d^{-\frac{2}{x}} - \frac{1}{4}d^{-\frac{3}{x}} + \frac{1}{5}d^{-\frac{4}{x}}) \times (2d^{-\frac{1}{x}} - 3d^{-\frac{2}{x}} + 4d^{-\frac{3}{x}})$.
43. $\left(\frac{5x^{-\frac{3}{2}}}{4y^{-\frac{3}{2}}} + \frac{3a^{-\frac{1}{2}}}{2b^{-\frac{1}{2}}}\right) \times \left(\frac{5x^{-\frac{3}{2}}}{4y^{-\frac{3}{2}}} - \frac{3a^{-\frac{1}{2}}}{2b^{-\frac{1}{2}}}\right)$.
44. $(2a^{-1}b^{-\frac{1}{2}} - 3a^{-\frac{1}{2}}b^{-\frac{1}{2}})^{2}$.

Perform the following divisions:

45. $e^{-\frac{1}{2}} \div e^{-\frac{1}{2}}$.
46. $h^{-\frac{3}{2}} \div h^{-\frac{1}{2}}$.
47. $k^{-\frac{3}{2}} \div k^{-\frac{1}{2}}$.
48. $l^{-\frac{3}{2}} \div l^{-\frac{1}{2}}$.
49. $g^{-\frac{3}{2}} \div g^{-\frac{3}{2}}$.
50. $a^{\frac{3}{2}} \div a^{-\frac{1}{2}}$.
51. $b^{-\frac{3}{2}} \div b^{\frac{3}{2}}$.
52. $m^{-\frac{1}{2}} \div m^{-\frac{1}{2}}$.
53. $p^{-\frac{1}{m}} \div p^{\frac{1}{n}}$.
54. $q^{\frac{1}{a}} \div q^{-\frac{1}{b}}$.
55. $t^{-\frac{m}{n}} \div t^{-\frac{1}{a}}$.
56. $v^{-\frac{1}{p}} \div v^{\frac{a}{b}}$.
57. $x^{-\frac{m}{n}} \div x^{-\frac{b}{2m}}$.
58. $z^{-a} \div z^{\frac{p}{q}}$.
59. $y^{-1} \div y^{-\frac{m}{n}}$.
60. $c^{-\frac{2}{m}} \div c^{-\frac{a}{m}}$.
61. $d^{-\frac{a}{m}} \div d^{-\frac{b}{2m}}$.
62. $e^{\frac{m}{n}} \div e^{-\frac{1}{a}}$.
63. $ab^{-\frac{p}{q}} \div a^{-\frac{1}{2}}b^{\frac{2p}{q}}$.
64. $\dfrac{1}{a^{-\frac{1}{2}}} \div \dfrac{3b^{-\frac{1}{2}}}{a^{-\frac{3}{2}}}$.

65. $(x^{\frac{3}{2}} - xy^{\frac{1}{2}} + x^{\frac{1}{2}}y - y^{\frac{3}{2}}) \div (x^{\frac{1}{2}} - y^{\frac{1}{2}})$.

66. $(x^{\frac{3}{2}} - x^2 - 4x^{\frac{3}{2}} + 6x - 2x^{\frac{1}{2}}) \div (x^{\frac{3}{2}} - 4x^{\frac{1}{2}} + 2)$.

67. $(\frac{2}{5}x^{\frac{7}{2}} + \frac{1}{2}x^{\frac{5}{2}} - \frac{2}{3}x^{\frac{7}{2}} - \frac{3}{5}x^{\frac{7}{2}} + x^{\frac{3}{2}} - \frac{3}{10}x^{\frac{3}{2}}) \div (x^{\frac{5}{2}} - \frac{3}{5}x^{\frac{3}{2}})$.

68. $(x + y + z - 3x^{\frac{1}{3}}y^{\frac{1}{3}}z^{\frac{1}{3}}) \div (x^{\frac{1}{3}} + y^{\frac{1}{3}} + z^{\frac{1}{3}})$.

Ex. 41.

Powers of Binomials.

Expand:

1. $(a+b)^3$.
2. $(m+n)^5$.
3. $(c+d)^8$.
4. $(h+l)^{10}$.
5. $(x+y)^{11}$.
6. $(y-z)^7$.
7. $(x-a)^9$.
8. $(y-b)^3$.
9. $(c-d)^5$.
10. $(m-a)^8$.
11. $(1+a)^4$.
12. $(a-1)^4$.
13. $(1+d)^6$.
14. $(1-h)^8$.
15. $(1+x)^{10}$.
16. $(2+a)^5$.
17. $(3-a)^6$.
18. $(m-2)^9$.
19. $(x-5)^3$.
20. $(y-3)^5$.
21. $(a^3+b^2)^7$.
22. $(m^5-n^2)^8$.
23. $(x^3+y^4)^5$.
24. $(a^4-y^3)^8$.
25. $(a-b^2)^7$.
26. $\left(\frac{a}{b}+\frac{x}{y}\right)^2$.
27. $\left(\frac{a}{c}-\frac{d}{x}\right)^3$.
28. $\left(\frac{m}{n}+\frac{p}{q}\right)^5$.
29. $\left(\frac{m}{p}-\frac{n}{q}\right)^6$.
30. $\left(\frac{1}{a}-\frac{b}{c}\right)^4$.
31. $(m^{\frac{3}{2}} - n^{\frac{1}{2}})^5$.
32. $(a^{\frac{2}{3}} + b^{\frac{2}{3}})^6$.
33. $(x^{\frac{1}{2}} + y^{\frac{1}{2}})^7$.
34. $(x^{\frac{1}{2}} - z^{\frac{1}{2}})^4$.
35. $(a^{\frac{1}{2}} + b^{\frac{1}{2}})^5$.
36. $(a^{-1} + b^{-2})^5$.
37. $(a^{-3} + x^{-1})^6$.
38. $(m^{-4} - n^{-1})^7$.
39. $(d^{-2} - x^2)^6$.
40. $(c^{-5} + z)^4$.
41. $(m^{-\frac{1}{2}} + n^{-\frac{1}{2}})^5$.
42. $(n^{-\frac{2}{3}} + p^{-\frac{1}{2}})^5$.
43. $(n^{-\frac{2}{3}} - p^{-\frac{1}{2}})^4$.
44. $(a^{-\frac{1}{2}} + b^{-\frac{1}{2}})^5$.
45. $(x^{-\frac{1}{2}} + y)^3$.
46. $(y^{-2} - z^{-\frac{1}{2}})^6$.
47. $(u^{-\frac{2}{3}} + v^3)^4$.
48. $(1^{-2} - z^{-\frac{1}{2}})^5$.
49. $(2x+y)^4$.
50. $(3y-z)^5$.
51. $(4z-u)^6$.
52. $(2a-b^2)^7$.
53. $(m^2+3b)^6$.

54. $(a-2c)^5$.
55. $(x^3-3d)^4$.
56. $(u^{-\frac{1}{2}}-4e)^3$.
57. $(2a^2-3b)^7$.
58. $(3x^{-1}+2y^{\frac{1}{2}})^6$.
59. $(2x+3z^{-\frac{1}{2}})^8$.
60. $\left(\frac{a}{2}-1\frac{1}{2}d^{-1}\right)^5$.
61. $(5m^{-2}+\frac{1}{5}n^{-1})^5$.
62. $\left(\frac{2a}{b}-\frac{3c}{d}\right)^6$.
63. $\left(\frac{1}{2a}+\frac{3}{b}\right)^5$.
64. $\left(\frac{2}{a^{-1}}+\frac{d^{-2}}{c^{-3}}\right)^4$.
65. $\left(\frac{x}{2}-\frac{y^{\frac{1}{3}}}{3}\right)^7$.
66. $\left(\frac{a}{3b}+\frac{2m^{\frac{1}{2}}}{x}\right)^5$.
67. $\left(1-\frac{3y}{4}\right)^6$.
68. $\left(\frac{3a}{b}-\frac{2a^2}{x}\right)^7$.
69. $(5m^x-\frac{1}{8}n^{2y})^4$.
70. $(3b^{m+1}+2d^{n-1})^5$.
71. $(4x^{a-3}-3y^{2-b})^4$.
72. $(5m^{2x-1}+\frac{3}{4}n^{y-2})^3$.
73. $(2a^3b-\frac{5}{8}mb^3)^3$.
74. $(\frac{1}{4}a^mb^x-\frac{1}{8}a^{m+1}b^{x-1})^4$.
75. $(2a^2b^m+\frac{1}{2}a^mb^2)^5$.

Ex. 42.

Transformation of Radicals.

$a=\sqrt[m]{a^m}$. $\qquad a\sqrt[m]{b}=\sqrt[m]{a^m}\times\sqrt[m]{b}=\sqrt[m]{a^mb}$.

$\sqrt[m]{ab}=\sqrt[m]{a}\times\sqrt[m]{b}$. $\qquad \sqrt[p]{a^pb}=a\sqrt[p]{b}$.

Express as entire surds:

1. $2\sqrt{2}$.
2. $5\sqrt{7}$.
3. $6\sqrt{5}$.
4. $2\sqrt{21}$.
5. $3\sqrt[3]{2}$.
6. $7\sqrt[3]{3}$.
7. $8\sqrt[3]{5}$.
8. $4\sqrt[4]{4}$.
9. $2\sqrt[5]{3}$.
10. $3\sqrt[7]{2}$.
11. $10\sqrt[3]{4}$.
12. $9\sqrt[3]{10}$.
13. $8\sqrt{41}$.
14. $a\sqrt{5}$.
15. $x\sqrt[4]{2}$.
16. $y\sqrt[5]{6}$.
17. $z\sqrt[3]{7}$.
18. $(-b)\sqrt[4]{6}$.
19. $(-d)\sqrt[5]{7}$.
20. $-m\sqrt[6]{10}$.
21. $-n\sqrt[3]{15}$.
22. $-3\sqrt{a}$.
23. $-5\sqrt[3]{b}$.
24. $-2\sqrt[5]{x}$.
25. $-4\sqrt[4]{y}$.
26. $3\sqrt{bx}$.
27. $a\sqrt{b}$.
28. $m^2\sqrt{yz^3}$.
29. $n^3\sqrt[5]{xy^3}$.
30. $-a^3\sqrt[4]{m^2n}$.
31. $-x^5\sqrt[3]{y^4z}$.
32. $\frac{1}{2}\sqrt{a}$.
33. $\frac{2}{3}\sqrt[3]{m^2}$.
34. $\sqrt{8}$.

Express as mixed surds:

35. $\sqrt{12}$.

36. $\sqrt{28}$.

37. $\sqrt{50}$.

38. $\sqrt{72}$.

39. $\sqrt[3]{72}$.

40. $\sqrt[3]{500}$.

41. $\sqrt[3]{108}$.

42. $\sqrt[3]{192}$.

43. $\sqrt[3]{56}$.

44. $\sqrt[5]{160}$.

45. $\sqrt[5]{2048}$.

46. $\sqrt[6]{3645}$.

47. $\sqrt[4]{567}$.

48. $\sqrt[4]{112}$.

49. $2\sqrt{405}$.

50. $3\sqrt[3]{864}$.

51. $5\sqrt[4]{144}$.

52. $\sqrt{a^2b}$.

53. $\sqrt[3]{m^3n^3}$.

54. $\sqrt[4]{b^6a^3}$.

55. $\sqrt[4]{b^4c}$.

56. $\sqrt[5]{c^{15}x^3}$.

57. $\sqrt[3]{x^5y^4}$.

58. $\sqrt{y^4z^3}$.

59. $\sqrt[3]{m^3n^7}$.

60. $\sqrt[4]{x^5y^6}$.

61. $\sqrt[3]{u^4v^{10}}$.

62. $\sqrt[4]{y^7z^5}$.

63. $\sqrt[10]{a^{12}b^{15}}$.

64. $\sqrt{4a^2b}$.

65. $\sqrt{25m^4x}$.

66. $\sqrt{36a^3b^2}$.

67. $\sqrt[3]{64x^6y}$.

68. $\sqrt[4]{81m^6n^2}$.

69. $\sqrt[6]{64a^3b^6}$.

70. $2\sqrt{80c^4d}$.

71. $\sqrt{\dfrac{4a^3b}{9}}$.

72. $\sqrt[3]{\dfrac{125m^3}{216n^5}}$.

73. $\sqrt{\dfrac{a^5}{4}}$.

74. $\sqrt[5]{\dfrac{m^6n}{243}}$.

75. $\sqrt[4]{\dfrac{x^5y^8}{1296}}$.

76. $\sqrt{\dfrac{(m^2-2mn+n^2)y}{49}}$.

77. $\sqrt[3]{\dfrac{(x^3-3x^2+3x-1)z}{512}}$.

78. $\sqrt{\dfrac{(x^2-y^2)^3}{4(x+y)}}$.

Ex. 43.

Reduction of Radicals.

$$\sqrt[m]{a^p} = \sqrt[mn]{a^{pn}}; \quad \sqrt[m]{a^n} = \sqrt[\frac{m}{p}]{a^{\frac{n}{p}}}.$$

$$(\sqrt{a}+\sqrt{b}) = \sqrt{a+b+2\sqrt{ab}}.$$

$$\sqrt{a}-\sqrt{b} = \sqrt{a+b-2\sqrt{ab}}.$$

$$\sqrt{a}+b\sqrt{-1} = \sqrt{a-b^2+2b\sqrt{a}\sqrt{-1}}.$$

$$\sqrt{a}-b\sqrt{-1} = \sqrt{a-b^2-2b\sqrt{a}\sqrt{-1}}.$$

Reduce the following radicals to the least common index possible:

1. $\sqrt{3}$; $\sqrt[6]{4}$.
2. $\sqrt[4]{6}$; $\sqrt{7}$.
3. $\sqrt{5}$; $\sqrt[3]{4}$.
4. $\sqrt[3]{2}$; $\sqrt[4]{3}$.
5. $\sqrt[5]{8}$; $\sqrt{2}$.
6. $\sqrt[3]{4}$; $\sqrt[4]{8}$.
7. $\sqrt{2}$; $\sqrt[3]{5}$; $\sqrt[4]{6}$.
8. $\sqrt[4]{7}$; $\sqrt[5]{5}$; $\sqrt[10]{120}$.

Change to similar radicals:

9. $\sqrt{3}$; $\sqrt{12}$.
10. $\sqrt{5}$; $\sqrt{20}$.
11. $\sqrt{63}$; $\sqrt{7}$.
12. $\sqrt{72}$; $\sqrt{8}$.
13. $3\sqrt{11}$; $\sqrt{176}$.
14. $\sqrt[3]{54}$; $2\sqrt[3]{2}$.
15. $\sqrt[3]{72}$; $\sqrt[3]{243}$.
16. $\sqrt[4]{5}$; $\sqrt[4]{405}$.
17. $\sqrt{18}$; $\sqrt{128}$; $\sqrt{32}$.
18. $7\sqrt[3]{54}$; $3\sqrt[3]{16}$; $\sqrt[3]{432}$.
19. $\sqrt{27}$; $2\sqrt{48}$; $3\sqrt{108}$.
20. $\sqrt[3]{128}$; $\sqrt[3]{686}$; $\sqrt[3]{16}$.
21. $\sqrt[4]{162}$; $3\sqrt[4]{32}$; $\sqrt[4]{2592}$.
22. $\sqrt{\frac{4}{5}}$; $2\sqrt{\frac{25}{8}}$.
23. $\frac{1}{2}\sqrt{\frac{2}{5}}$; $\sqrt{\frac{2}{45}}$.
24. $3\sqrt{\frac{50}{147}}$; $\sqrt{\frac{2}{363}}$.
25. $\frac{1}{4}\sqrt{0.2}$; $\frac{1}{5}\sqrt{5}$.
26. $\sqrt{20}$; $3\sqrt{\frac{1}{5}}$; $4\sqrt{125}$; $7\sqrt{\frac{8}{5}}$; $\sqrt[4]{\frac{25}{16}}$; $3\sqrt{\frac{9}{80}}$.
27. $\sqrt[3]{\frac{5}{8}}$; $\sqrt[3]{\frac{9}{8}}$; $\sqrt[3]{\frac{72}{343}}$; $\sqrt[3]{\frac{5}{8}}$; $2\sqrt[3]{\frac{3}{8}}$.
28. $\sqrt[5]{\frac{3}{16}}$; $\sqrt[5]{192}$; $\sqrt[5]{\frac{3}{512}}$; $\sqrt[5]{\frac{18}{81250}}$; $\sqrt[5]{\frac{2}{81}}$.

29. $\sqrt{\frac{1}{2}+\frac{3}{4}}$; $\sqrt{\frac{3}{5}-\frac{1}{3}}$.

30. $\sqrt{\frac{1}{4}-\frac{1}{5}}$; $\sqrt{\frac{4}{5}-\frac{7}{9}}$.

31. $\sqrt{\frac{13}{18}+\frac{1}{36}}$; $\sqrt{\frac{1}{27}+\frac{5}{9}}$.

32. $\sqrt[3]{\frac{12}{125}-\frac{1}{25}}$; $\sqrt[3]{\frac{47}{8}-\frac{1}{2}}$.

33. $\sqrt[6]{a^7b}$; $\sqrt[6]{a^{13}b^7}$.

34. $\sqrt[3]{27m^4}$; $\sqrt[3]{8m^7}$.

35. $\sqrt[3]{x^5-3x^4}$; $\sqrt[3]{\frac{x-3}{x^2}}$.

36. $\sqrt{\frac{a^4c}{b^3}}$; $\sqrt{\frac{a^2c^3}{bx^2}}$; $\sqrt{\frac{a^2cx^2}{by^3}}$.

37. $\sqrt{\frac{a^3x}{b^3}}$; $\sqrt{\frac{ax^5}{b^5}}$; $\sqrt{\frac{x^3}{ab}}$.

38. $\sqrt[4]{\frac{1}{a^3}}$; $\sqrt[4]{\frac{b^4}{a^3}}$; $\sqrt[4]{a}$; $\sqrt[4]{\frac{a}{c^4x^8}}$.

39. $\sqrt{1-\frac{1}{a^2}}$; $\sqrt{\frac{a-1}{a+1}}$.

40. $\sqrt{\left(\frac{1}{b}-a\right)}$; $\sqrt{\frac{bd^2-ab^2d^2}{c^2}}$.

41. $\sqrt{\left(\frac{a^2-b^2}{(a+b)}\right)^3}$; $\sqrt{\frac{(a^2+b^2)^2}{a-b}}$.

42. $\sqrt[3]{(x-b^2)^2}$; $\sqrt[3]{\frac{(y-z)^3}{x-b^2}}$; $\sqrt[3]{\frac{5y^5}{x-b^2}}$.

43. $\sqrt{\frac{m^3}{1-2x+x^2}}$; $\sqrt{\frac{4m-8mx+4mx^2}{c^2n^2}}$.

Change to a single radical:

44. $\sqrt{2}+\sqrt{18}$. 46. $\sqrt{50}+\sqrt{18}$. 48. $\sqrt{6}+\sqrt{24}$.

45. $\sqrt{12}+\sqrt{75}$. 47. $\sqrt{45}+\sqrt{245}$. 49. $\sqrt{5}+\sqrt{45}$.

50. $\sqrt{7}+\sqrt{28}$. 51. $\sqrt{11}+\sqrt{44}$.

Which is the greater:

52. $\sqrt{7}+\sqrt{10}$ or $\sqrt{3}+\sqrt{19}$?

53. $\sqrt{8}+\sqrt{12}$ or $\sqrt{2}+\sqrt{20}$?

54. $\sqrt{10}+\sqrt{8}+\sqrt{6}$ or $\sqrt{2}+\sqrt{3}+\sqrt{24}$?

Ex. 44.

Addition and Subtraction of Radicals.

Perform the following operations:

1. $2\sqrt{3}+3\sqrt{3}$.
2. $5\sqrt{7}+3\sqrt{7}+2\sqrt{7}$.
3. $8\sqrt{10}+3\sqrt{10}+2\sqrt{10}$.
4. $7\sqrt{2}-3\sqrt{2}$.
5. $4\sqrt{11}+3\sqrt{11}-5\sqrt{11}$.
6. $2\sqrt{3}-5\sqrt{3}+9\sqrt{3}$.
7. $4\sqrt[3]{3}-3\sqrt[3]{3}+2\sqrt[3]{3}$.
8. $11\sqrt{13}-4\sqrt{13}$.
9. $7\sqrt{2}-\sqrt{18}$.
10. $5\sqrt[3]{4}+2\sqrt[3]{32}-\sqrt[3]{108}$.
11. $5\sqrt{2}+2\sqrt{2}-4\sqrt{2}$.
12. $3\sqrt[5]{2}+4\sqrt[5]{2}-\sqrt[5]{64}$.
13. $\frac{1}{2}\sqrt[3]{5}+2\frac{1}{2}\sqrt[3]{5}+\frac{1}{4}\sqrt[3]{40}$.
14. $3\sqrt[4]{3}-5\sqrt[4]{48}+\sqrt[4]{243}$.
15. $2\sqrt{175}-3\sqrt{63}+5\sqrt{28}$.
16. $\sqrt{2}+3\sqrt{32}+\frac{1}{2}\sqrt{128}-6\sqrt{18}$.
17. $\sqrt{75}+\sqrt{48}-\sqrt{147}+\sqrt{300}$.
18. $20\sqrt{245}-\sqrt{5}+\sqrt{125}-2\frac{1}{2}\sqrt{180}$.
19. $2\sqrt{20}+\frac{1}{2}\sqrt{12}-2\sqrt{27}+5\sqrt{45}-9\sqrt{12}$.
20. $3\sqrt{7}+2\sqrt{54}-5\sqrt{63}+3\sqrt{150}+4\sqrt{63}+7\sqrt{24}$.
21. $7\sqrt{25}+4\sqrt{45}-\sqrt{9}-2\sqrt{80}+\sqrt{20}-4\sqrt{64}$.
22. $\sqrt{275}+\sqrt{1300}-\sqrt{44}+\sqrt{52}-2\sqrt{99}-2\sqrt{208}$.
23. $\sqrt{\frac{45}{4}}+\frac{3}{\sqrt{8}}-\sqrt{20}-\frac{\sqrt{245}}{6}-5\sqrt{\frac{1}{18}}-\sqrt{\frac{49}{2}}$.
24. $\sqrt[3]{\frac{4}{27}}-\sqrt[3]{\frac{1}{2}}-3\sqrt[3]{256}+\sqrt[3]{625}+\sqrt[3]{\frac{32}{27}}-\sqrt[3]{\frac{125}{64}}$.
25. $\sqrt{9}+\sqrt{20}-\sqrt{\frac{1}{16}}-\sqrt{\frac{9}{5}}+\sqrt{\frac{9}{16}}+\sqrt{\frac{25}{81}}+\sqrt{\frac{20}{9}}$.
26. $\sqrt[3]{54}+\sqrt{\frac{1}{2}}-\sqrt[3]{250}-\frac{3}{4}\sqrt{\frac{2}{3}}+0.5\sqrt{1\frac{2}{3}}+\sqrt[3]{6\frac{3}{4}}$.
27. $2\sqrt{\frac{5}{8}}+\sqrt{60}-\sqrt{15}+\sqrt{\frac{3}{8}}+\sqrt{\frac{4}{15}}$.

28. $3\frac{1}{2}\sqrt{24} - 5\frac{3}{4}\sqrt{54} + 13\frac{1}{3}\sqrt{99} + 2\frac{1}{2}\frac{7}{4}\sqrt{216} - 21\sqrt{44}$.

29. $\frac{3}{2}\sqrt[3]{\frac{2}{49}} + 0.8\sqrt{\frac{5}{8}} - \frac{1}{15}\sqrt{96} + 1.5\sqrt[3]{\frac{7}{4}} - \frac{11}{28}\sqrt[3]{1750} + 8\sqrt{\frac{3}{2}}$.

30. $\sqrt{a} - \frac{1}{2}\sqrt{a} + \frac{3}{2}\sqrt{a}$.

31. $\sqrt[3]{a^3} + \frac{1}{2}\sqrt[3]{a^3} - 3\sqrt[3]{27a^3}$.

32. $\sqrt{a^3} + b\sqrt{a} - 3\sqrt{a}$.

33. $\sqrt{25b} + 2\sqrt{9b} - 3\sqrt{4b}$.

34. $\sqrt[3]{27c^4} - \sqrt[3]{8c^4} + \sqrt[3]{125c}$.

35. $\sqrt[5]{a^5b} - \sqrt[5]{b^6} + \sqrt[5]{32b}$.

36. $\sqrt{a^4x} + \sqrt{b^4x} - \sqrt{4a^2b^2x}$.

37. $\sqrt{4x^2y^2z} + \sqrt{y^4z} + \sqrt{x^4z}$.

38. $\sqrt{a^2b^2c} - a\sqrt{4c} + b\sqrt{a^2c}$.

39. $\sqrt[4]{81a^5} - \sqrt[4]{16a} + \sqrt[4]{256a^5}$.

40. $\sqrt[3]{27m^4} - \sqrt[3]{125m} + \sqrt[3]{216m}$.

41. $\sqrt{8a} - \sqrt{50a^3} - 3\sqrt{18a}$.

42. $6a\sqrt{63ab^3} - 3\sqrt{112a^3b^3} + 2ab\sqrt{343ab} - 5b\sqrt{28a^3b}$.

43. $3\sqrt{125m^3n^2} + n\sqrt{20m^3} - \sqrt{500m^3n^2} - m\sqrt{45mn^2}$.

44. $\sqrt{32a^4b^5} + 6\sqrt{72b} + 3\sqrt{128a^2b^3} - 4\sqrt{288a^2b^3}$.

45. $2\sqrt[3]{a^6b} - 3a^2\sqrt[3]{64b} + 5a\sqrt[3]{a^3b} + 2a^2\sqrt[3]{125b^4} + 3\sqrt[3]{27a^6b}$.

Ex. 45.

Multiplication of Radicals.

$\sqrt[n]{a} \times \sqrt[n]{b} = \sqrt[n]{ab}$. $\sqrt[x]{a^y} = \sqrt[xz]{a^{yz}}$.

$\sqrt[m]{a^p} \times \sqrt[n]{c^q} = \sqrt[mn]{a^{pn}} \times \sqrt[mn]{c^{mq}} = \sqrt[mn]{a^{pn}c^{mq}}$.

$(\sqrt[n]{a})^n = a$.

Perform the following multiplications:

1. $\sqrt{3} \times \sqrt{27}$.
2. $\sqrt{5} \times \sqrt{20}$.
3. $\sqrt{2} \times \sqrt{18}$.

4. $\sqrt[3]{3} \times \sqrt[3]{9}$.
5. $\sqrt[3]{2} \times \sqrt[3]{32}$.
6. $\sqrt[3]{16} \times \sqrt[3]{32}$.
7. $\sqrt[4]{27} \times \sqrt[4]{3}$.
8. $\sqrt[4]{108} \times \sqrt[4]{12}$.
9. $\sqrt[5]{4} \times \sqrt[5]{8}$.
10. $\sqrt[5]{27} \times \sqrt[5]{9}$.
11. $\sqrt{2} \times \sqrt{32}$.
12. $\sqrt[3]{7} \times \sqrt[3]{-49}$.
13. $\sqrt[6]{1331} \times \sqrt[6]{11} \times \sqrt[6]{121}$.
14. $\sqrt[5]{8} \times \sqrt[5]{-4}$.
15. $\sqrt[4]{9} \times \sqrt[4]{81} \times \sqrt[4]{729}$.
16. $\sqrt{6} \times \sqrt{12} \times \sqrt{72}$.
17. $\sqrt[6]{5^3} \times \sqrt[6]{5^7} \times \sqrt[6]{5^6}$.
18. $\sqrt[3]{-9^4} \times \sqrt[3]{9} \times \sqrt[3]{9^4}$.
19. $\sqrt[3]{-108} \times \sqrt[3]{50} \times \sqrt[3]{40}$.
20. $\sqrt[5]{7^5} \times \sqrt[5]{-112} \times \sqrt[5]{14}$.
21. $\sqrt{3} \times \sqrt{6}$.
22. $\sqrt[3]{3} \times \sqrt[3]{18}$.
23. $\sqrt[4]{6} \times \sqrt[4]{8}$.
24. $\sqrt[5]{54} \times \sqrt[5]{9}$.
25. $\sqrt[3]{81} \times \sqrt[3]{-45}$.
26. $2\sqrt{8} \times \sqrt{2}$.
27. $3\sqrt[3]{16} \times 5\sqrt[3]{4}$.
28. $\frac{1}{2}\sqrt[4]{8} \times \frac{3}{4}\sqrt[4]{2}$.
29. $\frac{2}{3}\sqrt[3]{18} \times \frac{3}{4}\sqrt[3]{-3}$.
30. $5\sqrt{2} \times 0.4\sqrt{72}$.
31. $(\sqrt{18} + 2\sqrt{72} - 3\sqrt{8} - \frac{1}{2}\sqrt{128})\sqrt{2}$.
32. $(\sqrt[3]{32} - \frac{1}{2}\sqrt[3]{864} + 3\sqrt[3]{4})\sqrt[3]{2}$.
33. $(\frac{1}{3}\sqrt{27} - \frac{1}{4}\sqrt{2187} + \frac{1}{8}\sqrt{432} - \sqrt{16{,}875})\sqrt{3}$.
34. $(3\sqrt[3]{24} + 4\sqrt[3]{81} - 5\sqrt[3]{375} + \frac{1}{2}\sqrt[3]{192})\sqrt[3]{2}$.
35. $(\sqrt{5 - \sqrt{4}})(\sqrt{5 + \sqrt{4}})$.
36. $(\sqrt{3 - \sqrt{2}})(\sqrt{3 + \sqrt{2}})$.
37. $(\sqrt{9 - \sqrt{17}})(\sqrt{9 + \sqrt{17}})$.
38. $(\sqrt{6 + \sqrt{11}})(\sqrt{6 - \sqrt{11}})$.
39. $(\sqrt{8 + \sqrt{15}})(\sqrt{8 - \sqrt{15}})$.
40. $(\sqrt[3]{10 - \sqrt{36}})(\sqrt[3]{10 + \sqrt{36}})$.

41. $(\sqrt[3]{9+\sqrt{17}})(\sqrt[3]{9-\sqrt{17}})$.

42. $(\sqrt[3]{12+\sqrt{19}})(\sqrt[3]{12-\sqrt{19}})$.

43. $(3+2\sqrt{5})(2-\sqrt{5})$.

44. $(8+3\sqrt{2})(2-\sqrt{2})$.

45. $(5+2\sqrt{3})(3-5\sqrt{3})$.

46. $(3-\sqrt{6})(6-3\sqrt{6})$.

47. $(2\sqrt{6}-3\sqrt{5})(\sqrt{3}+2\sqrt{2})$.

48. $(\sqrt{7}-\sqrt{3})(\sqrt{2}+\sqrt{5})$.

49. $(\sqrt[3]{9}-2\sqrt[3]{4})(4\sqrt[3]{3}+\sqrt[3]{2})$.

50. $(2\sqrt{30}-3\sqrt{5}+5\sqrt{3})(\sqrt{8}+\sqrt{3}-\sqrt{5})$.

51. $(3\sqrt{5}-2\sqrt{3}+4\sqrt{7})(3\sqrt{7}-4\sqrt{5}-5\sqrt{3})$.

52. $(2\sqrt{\frac{1}{2}}-3\sqrt{3}+4\sqrt{8})(3\sqrt{8}+\sqrt{18}-\sqrt{3})$.

53. $(3\sqrt{3}-4\sqrt{8}+5\sqrt{18})(5\sqrt{2}+\sqrt{3}+\sqrt{\frac{1}{2}})$.

54. $(4\sqrt{8}+\frac{1}{2}\sqrt{12}-\frac{1}{4}\sqrt{32})(8\sqrt{32}-4\sqrt{50}-2\sqrt{2})$.

55. $(\sqrt[3]{6}-\sqrt[3]{3}+\sqrt[3]{16})(\sqrt[3]{36}+\sqrt[3]{9}-\sqrt[3]{4})$.

56. $(\sqrt[3]{16}-2\sqrt[3]{2}+4\sqrt[3]{54})(5\sqrt[3]{4}+3\sqrt[3]{\frac{1}{2}}-2\sqrt[3]{2})$.

57. $(\sqrt[3]{9}-7\sqrt[3]{72}+6\sqrt[3]{1125})(3\sqrt[3]{2}+8\sqrt[3]{3}-4\sqrt[3]{\frac{1}{3}})$.

58. $(0.4\sqrt[3]{25}+0.2\sqrt[3]{200}-\frac{1}{2}\sqrt[3]{75})(2\sqrt[3]{135}-5\sqrt[3]{5}-10\sqrt[3]{15})$

59. $(2\sqrt{\frac{2}{3}}-8\sqrt{\frac{3}{8}}+3\sqrt{\frac{8}{2}})(3\sqrt{\frac{2}{3}}-\sqrt{12}-\sqrt{6})$.

60. $(2\sqrt{\frac{5}{6}}-4\sqrt{\frac{3}{2}}-7\sqrt{\frac{6}{5}})(3\sqrt{\frac{5}{6}}-5\sqrt{30}-2\sqrt{1\frac{5}{2}})$.

61. $(3\sqrt{2}+4\sqrt{5}-2\sqrt{3})^2$.

62. $(5\sqrt{6}-3\sqrt{3}-2\sqrt{24})^2$.

63. $(3\sqrt{5}-4\sqrt{45}+2\sqrt{20})^2$.

64. $(2\sqrt{12}+3\sqrt{3}+6\sqrt{\frac{1}{3}})^2$.

65. $(2\sqrt{7}-8\sqrt{28}-5\sqrt{63})^2$.

66. $(\frac{1}{2}\sqrt{24}-\frac{3}{4}\sqrt{6}+5\sqrt{3})^2$.

67. $(3\sqrt{x^5}-2\sqrt{x^3})^2$.

68. $(a\sqrt{y}+b\sqrt{y^3})^2$.

69. $(m^2\sqrt{ab^3}-m\sqrt{a^3b})^2$.

70. $(2p\sqrt{12z^5}-\frac{1}{2}q\sqrt{18z})^2$.

71. $\sqrt{5} \times \sqrt[3]{4}$.
72. $\sqrt{125} \times \sqrt[4]{36}$.
73. $\sqrt[3]{16} \times \sqrt[6]{250}$.
74. $\sqrt[4]{36} \times \sqrt[8]{1296}$.
75. $\sqrt[4]{64} \times \sqrt[8]{16}$.
76. $\sqrt[8]{20} \times \sqrt[9]{1000}$.
77. $\sqrt[3]{\frac{1}{2}} \times \sqrt[12]{256^3}$.
78. $\sqrt{15} \times \sqrt[6]{27 \times 125}$.
79. $\sqrt[5]{12} \times \sqrt[10]{96}$.
80. $3 \times \sqrt[3]{72}$.
81. $\sqrt[15]{27} \times \sqrt[4]{42}$.
82. $\sqrt[24]{256} \times \sqrt[5]{2}$.
83. $\sqrt[7]{7^5} \times \sqrt{5^6}$.
84. $\sqrt[6]{6^3} \times \sqrt[10]{42^7}$.
85. $\sqrt[5]{\frac{3}{8}} \times \sqrt[3]{\frac{2}{3}}$.
86. $\sqrt[4]{\frac{7}{9}} \times \sqrt[5]{\frac{15}{7}}$.
87. $\sqrt[3]{\frac{4}{81}} \times \sqrt[5]{\frac{27}{32}}$.
88. $\sqrt[9]{\frac{9}{28}} \times \sqrt[4]{\frac{2}{3}}$.
89. $\sqrt[10]{\frac{125}{16}} \times \sqrt[3]{\frac{4}{5}}$.

Ex. 46.

Division of Radicals.

Perform the following divisions:

1. $\sqrt{243} \div \sqrt{3}$.
2. $\sqrt[3]{81} \div \sqrt{3}$.
3. $\sqrt[3]{56} \div \sqrt[3]{7}$.
4. $\sqrt[3]{500} \div \sqrt[3]{4}$.
5. $\sqrt[3]{1080} \div \sqrt[3]{5}$.
6. $\frac{\sqrt[3]{72}}{\sqrt[3]{9}}$.
7. $\sqrt[4]{80} \div \sqrt[4]{5}$.
8. $\sqrt[4]{243} \div \sqrt[4]{3}$.
9. $\sqrt[4]{4375} \div \sqrt[4]{7}$.
10. $\sqrt[4]{2048} \div \sqrt[4]{8}$.
11. $\sqrt{\frac{1}{2}} \div \sqrt{\frac{2}{9}}$.
12. $\sqrt{\frac{5}{6}} \div \sqrt{\frac{3}{5}}$.
13. $\sqrt{\frac{7}{12}} \div \sqrt{\frac{3}{7}}$.
14. $\sqrt{\frac{2}{3}} \div \sqrt{\frac{1}{2}}$.
15. $\sqrt{\frac{12}{35}} \div \sqrt{\frac{21}{5}}$.
16. $\sqrt{\frac{14}{15}} \div \sqrt{\frac{24}{35}}$.
17. $\sqrt{\frac{22}{27}} \div \sqrt{\frac{6}{11}}$.
18. $\sqrt{\frac{52}{55}} \div \sqrt{\frac{17}{65}}$.
19. $(3\sqrt{6} + 45\sqrt{2}) \div 3\sqrt{3}$.
20. $(42\sqrt{5} - 30\sqrt{3}) \div 2\sqrt{15}$.
21. $(84\sqrt{15} + 168\sqrt{6}) \div 3\sqrt{21}$.
22. $(15\sqrt{105} - 36\sqrt{10} + 30\sqrt{3}) \div 3\sqrt{15}$.

23. $(56\sqrt{30} - 84\sqrt{10} + 100\sqrt{14}) \div 4\sqrt{35}$.

24. $(36\sqrt[3]{4} - 30\sqrt[3]{16} + 30\sqrt[3]{20}) \div 8\sqrt[3]{20}$.

25. $(50\sqrt[3]{18} + 18\sqrt[3]{20} - 48\sqrt[3]{5}) \div 2\sqrt[3]{30}$.

26. $(2 - \sqrt{2}) \div (2 + \sqrt{2})$.

27. $\sqrt{3a} \div \sqrt{a}$.

28. $\sqrt{2x} \div \sqrt{2}$.

29. $\sqrt{10y} \div \sqrt{5y}$.

30. $\sqrt{12m} \div \sqrt{6}$.

31. $\sqrt{5b} \div \sqrt{2b}$.

32. $\sqrt{a^5} \div \sqrt{a^3}$.

33. $\sqrt{3a^7} \div \sqrt{a^3}$.

34. $\sqrt{8c^9} \div \sqrt{2c^5}$.

35. $\sqrt{\frac{a^3}{b^5}} \div \sqrt{\frac{a}{b^3}}$.

36. $\sqrt{\frac{3}{a^3}} \div \sqrt{\frac{6}{a}}$.

37. $\sqrt[3]{54a^7} \div \sqrt[3]{2a}$.

38. $\sqrt[3]{96n^8} \div \sqrt[3]{6n^5}$.

39. $\sqrt[3]{49x^6} \div \sqrt[3]{\frac{1}{7}x}$.

40. $\sqrt[3]{2y} \div \sqrt[3]{\frac{1}{4y^2}}$.

41. $\sqrt[3]{a^2x} \div \sqrt[3]{ax^5}$.

42. $\sqrt[4]{125a^7} \div \sqrt[4]{\frac{a^3}{5}}$.

43. $\sqrt[4]{2\frac{7}{2}y} \div \sqrt[4]{\frac{8y^5}{3}}$.

44. $\sqrt{\frac{54a^3}{y}} \div \sqrt[4]{\frac{8y^5}{9a}}$.

45. $\sqrt[4]{\frac{2}{5}m^3} \div \sqrt[4]{\frac{5m}{8}}$.

46. $(a - b) \div (\sqrt{a} - \sqrt{b})$.

48. $(b - 1) \div (\sqrt{b} - 1)$.

47. $(1 - a) \div (1 - \sqrt{a})$.

49. $1 \div \sqrt{\frac{x + 2y}{x^3 - 3xy^2 + 2y^3}}$.

50. $(acx^2\sqrt{y} - bcy\sqrt{x}) \div c\sqrt{xy}$.

51. $(acy^2\sqrt[3]{z^2} - adz\sqrt[3]{y}) \div a\sqrt[3]{yz^2}$.

52. $\sqrt{54} \div \sqrt[4]{36}$.

53. $\sqrt[6]{9} \div \sqrt{3}$.

54. $\sqrt[3]{49} \div \sqrt{7}$.

55. $\sqrt[3]{72} \div \sqrt{6}$.

56. $\sqrt[3]{12} \div \sqrt{6}$.

57. $\sqrt[4]{0.0324} \div \sqrt{0.5}$.

58. $\sqrt[6]{0.064} \div \sqrt{10}$.

59. $\sqrt[6]{10.4976} \div 2\sqrt{5}$.

60. $\sqrt{\frac{8}{45}} \div \sqrt[4]{6\frac{1}{4}}$.

61. $\sqrt{\frac{2}{5}} \div \sqrt[6]{3\frac{3}{8}}$.

62. $\sqrt[5]{\frac{4}{5}} \div 2\sqrt[10]{\frac{1}{400}}$.

63. $\sqrt{27} \div 3$.

64. $\sqrt{196} \div 2$.

65. $\sqrt{45} \div 3$.

66. $\sqrt{1452} \div 11$.

67. $6 \div \sqrt{24}$.

68. $8 \div \sqrt{72}$.

69. $7 \div \sqrt{63}$.

70. $30 \div \sqrt{210}$.

71. $6 \div (3\sqrt{3} - 5)$.

72. $4 \div (3\sqrt{2} - 4)$.

73. $\sqrt{a} \div \sqrt[3]{a^2}$.

74. $\sqrt[3]{b^2} \div \sqrt[5]{b^3}$.

75. $\sqrt[3]{m^5} \div \sqrt[4]{m^7}$.

76. $\sqrt{x^3} \div \sqrt[3]{x^2}$.

77. $\sqrt[3]{2x} \div \sqrt{x^3}$.

78. $\sqrt[5]{y^4} \div \sqrt[7]{2y^3}$.

79. $\sqrt[6]{\frac{z^9}{27}} \div \sqrt{6z^5}$.

80. $\sqrt[11]{a^5b^9} \div \sqrt[7]{a^4b^3}$.

81. $\sqrt[7]{\frac{a^2}{c^3}} \div \sqrt[12]{\frac{a^2}{c^8}}$.

82. $a^2 \div \sqrt{a}$.

83. $b^3 \div \sqrt{b}$.

84. $az^5 \div \sqrt{a^2z}$.

85. $3ny^4 \div \sqrt{6n^2y}$.

86. $\sqrt[3]{25m^4x} \div 5mx^2$.

87. $\sqrt[4]{8x^3y^2} \div 2x^2y^3$.

88. $\sqrt[6]{1600a^{22}x^{16}} \div 5a^2x^4$.

89. $(x+y) \div \frac{1}{3}\sqrt{x^2-y^2}$.

90. $(3mn\sqrt[3]{n} + 4n^2\sqrt{m}) \div 6\sqrt[4]{mn^6}$.

Perform the following operations:

91. $(\sqrt[-2]{25})$.

92. $\sqrt[-3]{\frac{1}{64}}$.

93. $\sqrt[-6]{\frac{64}{15625}}$.

94. $\sqrt[-4]{36}$.

95. $\sqrt[1]{17}$.

96. $\sqrt[-1]{6}$.

97. $\sqrt[-1]{\frac{1}{4}}$.

98. $\sqrt[-5]{243}$.

99. $\sqrt[-2]{a}$.

100. $\sqrt[-3]{m^6}$.

101. $\sqrt[-1]{a}$.

102. $\sqrt[-6]{\frac{m^{14}}{n^{18}}}$.

103. $\sqrt[-4]{\frac{a^6}{b^{10}}}$.

104. $\sqrt[-2]{a^3} \times \sqrt[2]{a^5}$.

105. $\sqrt[3]{m^5} \times \sqrt[-5]{m^3}$.

106. $\sqrt[-3]{x^5} \times \sqrt[-2]{x^7}$.

107. $\sqrt[-5]{\frac{1}{a^2}} \times \sqrt[-2]{a^3}$.

108. $\frac{\sqrt{a^3}}{\sqrt[-2]{a^5}}$.

109. $\frac{\sqrt[-3]{m^2}}{\sqrt[3]{m^5}}$.

110. $\frac{\sqrt{a}}{\sqrt[-3]{a}}$.

111. $\frac{\sqrt{x^5}}{\sqrt[-5]{x}}$.

112. $\frac{\sqrt[-3]{a^2}}{\sqrt[-5]{a^3}}$.

113. $\sqrt[\frac{3}{2}]{4}$.

114. $\sqrt[\frac{3}{2}]{12}$.

115. $\sqrt[\frac{3}{2}]{27}$.

Ex. 47.

Simplification of Radicals.

Perform the following operations:

1. $(\sqrt{a^3})^3$.
2. $(\sqrt[3]{m^2})^3$.
3. $(\sqrt[5]{n^3})^5$.
4. $(\sqrt[4]{x^3})^4$.
5. $(\sqrt[7]{y^3})^7$.
6. $(\sqrt[9]{ab^2})^9$.
7. $(\sqrt{m^3})^5$.
8. $(\sqrt[3]{n^2})^4$.
9. $(\sqrt[4]{y})^5$.
10. $(\sqrt[3]{b^2c})^4$.
11. $(\sqrt[5]{x^2z^3})^7$.
12. $(\sqrt[3]{2a^4})^2$.
13. $(\sqrt{\sqrt[3]{2\frac{14}{15}}})^9$.
14. $(\sqrt[3]{\sqrt{27a^6}})^7$.
15. $(\sqrt[m]{\sqrt[n]{a^{2m}c^m}})^n$.
16. $(\sqrt[5]{\sqrt[4]{16a^{17}}})^5$.
17. $(\sqrt[3]{ax^4})^6$.
18. $[\sqrt{(x+y)^3}]^4$.
19. $[\sqrt[5]{(x-y)^2}]^4$.
20. $[\sqrt[6]{(a^2-b^2)^4}]^1$.
21. $[\sqrt[8]{(1+a^2)^6}]^3$.
22. $[\sqrt[m]{(x^2-y^2)^{2m}}]^n$.
23. $(\sqrt[3]{a^3b}-\sqrt{m^3}+\sqrt[4]{ax^3})^2$.
24. $(\sqrt[x]{a^2}-\sqrt[x]{b})^2$.
25. $(\sqrt[x]{m}-\sqrt[x]{n}+\sqrt[x]{p^2})^2$.
26. $(-\sqrt{2a^3b})^2$.
27. $(-\sqrt[3]{xy^2})^3$.
28. $(-\sqrt[3]{x^2-y^2})^6$.
29. $(-\sqrt[3]{a^2-b^2})^9$.
30. $(-\sqrt{5})^2$.
31. $(-\sqrt[3]{4})^6$.
32. $(-\sqrt[4]{127})^4$.
33. $(-\sqrt[5]{131})^5$.
34. $(-\sqrt[3]{9})^4$.
35. $(-\sqrt[4]{8})^3$.
36. $(-\sqrt[7]{13})^{14}$.
37. $(-\sqrt{-\sqrt{6}})^4$.
38. $(\sqrt[8]{n})^4$.
39. $(\sqrt[3]{v^2})^3$.
40. $(\sqrt[5]{x^4})^{15}$.
41. $(\sqrt[4]{x^3})^4$.
42. $(\sqrt[3]{y^5})^{12}$.
43. $\sqrt{\sqrt[3]{a^2}}$.
44. $\sqrt[3]{\sqrt{b}}$.
45. $\sqrt[4]{\sqrt[3]{c}}$.
46. $\sqrt[5]{\sqrt[3]{2m}}$.
47. $\sqrt[4]{\sqrt{3a^2}}$.
48. $\sqrt[6]{\sqrt[3]{a^6}}$.
49. $\sqrt[5]{\sqrt[3]{m^5}}$.
50. $\sqrt[6]{\sqrt[7]{y^{14}}}$.
51. $\sqrt[5]{\sqrt[3]{a^{10}}}$.
52. $\sqrt[4]{\sqrt[3]{m^5}}$.
53. $\sqrt[3]{\sqrt{125}}$.
54. $\sqrt[4]{\sqrt[3]{729}}$.
55. $\sqrt[4]{\sqrt{(x-y)^8}}$.

56. $\sqrt[3]{\sqrt[4]{(a-b)^{36}}}$. 59. $\sqrt{\sqrt[5]{(1-x^3)^{60}}}$. 62. $\sqrt[11]{\sqrt[3]{m^{33}n^{22}}}$.

57. $\sqrt[5]{\sqrt[3]{(m+n)^{30}}}$. 60. $\sqrt[9]{\sqrt[3]{512x^{18}}}$. 63. $\sqrt[13]{6\sqrt[5]{m^{65}}}$.

58. $\sqrt[7]{\sqrt[3]{(3a-2b)^{14}}}$. 61. $\sqrt[3]{\sqrt[5]{32a^{45}}}$. 64. $\sqrt[7]{128\sqrt[5]{243a^{70}}}$.

Ex. 48.

RATIONALIZATION OF THE DENOMINATORS OF RADICALS.

In the following expressions make the denominators rational:

1. $\frac{2\sqrt{2}}{3\sqrt{2}-2\sqrt{3}}$.

2. $\frac{21\sqrt{3}}{4\sqrt{3}-3\sqrt{2}}$.

3. $\frac{120\sqrt{2}}{5\sqrt{6}+3\sqrt{10}}$.

4. $\frac{75\sqrt{14}}{8\sqrt{2}+2\sqrt{7}}$.

5. $\frac{1200\sqrt{10}}{12\sqrt{5}-4\sqrt{7\frac{1}{2}}}$.

6. $\frac{\sqrt{5}-\sqrt{3}}{\sqrt{5}+\sqrt{3}}$.

7. $\frac{\sqrt{11}+\sqrt{5}}{\sqrt{11}-\sqrt{5}}$.

8. $\frac{\sqrt{13}-\sqrt{10}}{\sqrt{13}+\sqrt{10}}$.

9. $\frac{\sqrt{8}+\sqrt{7}}{\sqrt{7}-\sqrt{2}}$.

10. $\frac{\sqrt{6}+\sqrt{10}}{\sqrt{10}-\sqrt{6}}$.

11. $\frac{3+5\sqrt{3}}{4-3\sqrt{2}}$.

12. $\frac{1-4\sqrt{5}}{6+2\sqrt{3}}$.

13. $\frac{7-3\sqrt{10}}{5+4\sqrt{5}}$.

14. $\frac{2+2\sqrt{2}}{6+5\sqrt{6}}$.

15. $\frac{7+3\sqrt{7}}{12-6\sqrt{11}}$.

16. $\frac{18}{4+3\sqrt{2}+5\sqrt{3}}$.

17. $\frac{1-\sqrt{2}+\sqrt{3}}{1+\sqrt{2}-\sqrt{3}}$.

18. $\frac{2+\sqrt{6}-\sqrt{2}}{2-\sqrt{6}+\sqrt{2}}$.

19. $\frac{3-\sqrt{5}-\sqrt{2}}{3+\sqrt{5}+\sqrt{2}}$.

20. $\frac{7-2\sqrt{3}+3\sqrt{2}}{3+3\sqrt{3}-2\sqrt{2}}$.

21. $\frac{2\sqrt{3}-3\sqrt{6}}{\sqrt{2}-\sqrt{3}+\sqrt{5}-\sqrt{6}}$.

Ex. 49.

Imaginary Expressions.

$\sqrt{-1} = i.$ $\quad i^{4n+1} = \sqrt{-1}.$

$(\sqrt{-1})^2 = i^2 = -1.$ $\quad i^{4n+2} = -1.$

$(\sqrt{-1})^3 = i^3 = -\sqrt{-1}.$ $\quad i^{4n+3} = -\sqrt{-1}.$

$(\sqrt{-1})^4 = i^4 = +1.$ $\quad i^{4n+4} = +1.$

Place the following quantities under the form $a\sqrt{-1}$.

1. $\sqrt{-4}$.
2. $\sqrt{-25}$.
3. $\sqrt{-81}$.
4. $\sqrt{-144}$.
5. $\sqrt{-a^2}$.
6. $\sqrt{-b^4}$.
7. $\sqrt[4]{-16}$.
8. $\sqrt[4]{-81}$.
9. $\sqrt[4]{-625}$.
10. $\sqrt[6]{-64}$.
11. $\sqrt[6]{-729}$.
12. $\sqrt[8]{-256}$.
13. $\sqrt[4]{-x^8}$.
14. $\sqrt[4]{-y^9}$.
15. $\sqrt[6]{-z^{18}}$.
16. $\sqrt[10]{-u^{20}}$.
17. $\sqrt{-\frac{1}{4}}$.
18. $\sqrt{-\frac{a^4}{b^2}}$.
19. $\sqrt{-x^2-y^2}$.
20. $\sqrt[4]{-(x^2+y^2)^2}$.
21. $\sqrt[6]{-(3x-2y^2)^{12}}$.
22. $\sqrt{-9x^6}$.
23. $\sqrt{-81m^6}$.

What is the value of:

24. $(\sqrt{-1})^6$; $(\sqrt{-1})^8$; $(\sqrt{-1})^{26}$; $(\sqrt{-1})^{16}$; $(\sqrt{-1})^{21}$; $(\sqrt{-1})^{27}$; $(\sqrt{-1})^{51}$?
25. i^7; i^{11}; i^{28}; i^{32}; i^{14}; i^{13}; i^{38}?

Add:

26. $\sqrt{-25}+\sqrt{-49}+\sqrt{-121}-\sqrt{-64}+\sqrt{-1}-\sqrt{-36}$.
27. $\sqrt{-a^4}+\sqrt{-a^2}-\sqrt{-4a^4}+\sqrt{-16a^4}-\sqrt{-81a^2}+\sqrt{-a^2}$.
28. $\sqrt{-(x-y)^2}$; $\sqrt{-(x^2-2xy+y^2)}$; $+\sqrt{-64x^2y^2}$.
29. $\sqrt{-25a^6}$; $\sqrt{-16a^6}$; $-\sqrt{-(a-1)^2a^6}$.
30. $\sqrt{-(m-n)^2}$; $-\sqrt{-(n-2m)^2}$; $\sqrt{-4m^2n^2}$; $-\sqrt{-4(m-n)^2}$.

31. $5+\sqrt{-16}$; $3-\sqrt{-4}$; $8+\sqrt{-4}$; $-(7+\sqrt{-36})$; $2-\sqrt{-1}$.

32. $a+bi$; $2a+3bi$; $\frac{1}{2}a-4bi$; $5a-\frac{1}{2}bi$; $4a-16bi$.

33. $3+2i$; $4-2i$; $7+3i$; $8-i$; $4+2bi$.

Multiply:

34. $\sqrt{-a}\times\sqrt{-a}$.

35. $\sqrt{-16}\times\sqrt{-9}$.

36. $\sqrt{-m}\times\sqrt{-n}$.

37. $\sqrt{-y}\times\sqrt{-1}$.

38. $\sqrt{25}\times\sqrt{-49}$.

39. $\sqrt{a^8}\times\sqrt{-a^4}$.

40. $\sqrt{-9y^2}\times\sqrt{-16z^6}$.

41. $\sqrt{-m^2}\times(-\sqrt{-9m^4})$.

42. $i\sqrt{-z^2}$.

43. $i\sqrt{-36}\times\sqrt{25}$.

44. $i\sqrt{-28}\times i\sqrt{-32}$.

45. $i^2\sqrt{-49}\times\sqrt{-121}$.

46. $\sqrt{-(a+b)}\times\sqrt{-(a-b)}$.

47. $\sqrt{x-y}\times\sqrt{y-x}$.

48. $\sqrt{y-5}\times\sqrt{5-y}$.

49. $i\sqrt{-9}\times\sqrt{2-y}$.

50. $(a+\sqrt{-1})(a-\sqrt{-1})$.

51. $(m+bi)(m-bi)$.

52. $(3+5i)(4-7i)$.

53. $(x+2i)(y-bi)$.

54. $(3x-8i)(2y+3i)$.

55. $(2a+3bi)(c-di)$.

56. $(4-i\sqrt{8})(3+i\sqrt{2})$.

57. $(m-3i\sqrt{b})(n+4i\sqrt{c})$.

58. $(\sqrt{5}+i\sqrt{6})(\sqrt{6}-i\sqrt{8})$.

Perform the following divisions:

59. $\frac{a}{\sqrt{-1}}$.

60. $\frac{b}{\sqrt{-b^2}}$.

61. $\frac{d}{\sqrt{-4}}$.

62. $\frac{\sqrt{-9}}{\sqrt{-3}}$.

63. $\frac{\sqrt{-m}}{\sqrt{-n}}$.

64. $\frac{\sqrt{-ax}}{\sqrt{-x}}$.

65. $\frac{x}{\sqrt{-x}}$.

66. $\frac{z}{\sqrt{-z^2}}$.

67. $\frac{m}{i\sqrt{m}}$.

68. $\frac{\sqrt{-x^3}}{\sqrt{-x}}$.

69. $\frac{\sqrt{-8}}{\sqrt{-2}}$.

70. $\frac{\sqrt{-10m^3}}{\sqrt{-5m}}$.

Ex. 50.

Solve: RADICAL EQUATIONS.

1. $2\sqrt{x+5}=\sqrt{28}$.
2. $3\sqrt{4x-8}=\sqrt{13x-3}$.
3. $\sqrt{x+9}=5\sqrt{x-3}$.
4. $4=2\sqrt{x}-3$.
5. $5-\sqrt{3y}=4$.
6. $9+4\sqrt{x}=11$.
7. $6-3\sqrt{x}=4$.
8. $7+2\sqrt[3]{3x}=5$.
9. $11-4\sqrt{5x}=9$.
10. $6+\sqrt{8x}=2$.
11. $\sqrt[3]{2x-3}=-3$.
12. $\sqrt{8y+17}=4$.
13. $\sqrt[3]{3x+7}=3$.
14. $\sqrt[5]{80x-43}=-3$.
15. $14+\sqrt[3]{4x-40}=10$.
16. $\sqrt{2x+8}=\sqrt{5x-2}$.
17. $\sqrt[3]{10y-4}=\sqrt[3]{7y+11}$.
18. $\sqrt[5]{2x-3}=\sqrt[5]{3x+2}$.
19. $2x-8=\sqrt{4x^2-12x+32}$.
20. $\sqrt[3]{3y+2}=\sqrt[6]{9y^2+10y-6}$.
21. $\sqrt{x-1}=\sqrt[3]{(x-1)^2(x+3)^2}$.
22. $\sqrt{(2x-5)}=\sqrt[6]{\frac{1}{8}(2x-5)^4}$.
23. $\sqrt{(3x-4)}=\sqrt[6]{(3x-4)^3(9x-6)}$.
24. $2\sqrt{y-2}=\sqrt[4]{32(y-2)^3}$.
25. $\sqrt[4]{9x^4+6x^3+5x^2-2x}=\sqrt{3x^2+x}$.
26. $\sqrt[6]{8x^3-60x^2+145x-115}=\sqrt{2x-5}$.
27. $\sqrt[3]{(3x^2-5x+2)(2x+4)}=\sqrt[6]{(3x^2-5x+2)^2(4x^2-5x+1)}$.
28. $\sqrt{(5x^5-x^4+2)(5x+3)}=\sqrt[4]{(5x^5-x^4+2)^2(25x^2+25x-6)}$.
29. $3\sqrt{16x+9}=12\sqrt{4x}-9$.
30. $\sqrt{\frac{15}{4}+x}=\frac{5}{2}+\sqrt{x}$.
31. $\sqrt{32+x}=16-\sqrt{x}$.

31. $5+\sqrt{-16}$; $3-\sqrt{-4}$; $8+\sqrt{-4}$; $-(7+\sqrt{-36})$; $2-\sqrt{-1}$.

32. $a+bi$; $2a+3bi$; $\frac{1}{2}a-4bi$; $5a-\frac{1}{2}bi$; $4a-16bi$.

33. $3+2i$; $4-2i$; $7+3i$; $8-i$; $4+2bi$.

Multiply:

34. $\sqrt{-a}\times\sqrt{-a}$.

35. $\sqrt{-16}\times\sqrt{-9}$.

36. $\sqrt{-m}\times\sqrt{-n}$.

37. $\sqrt{-y}\times\sqrt{-1}$.

38. $\sqrt{25}\times\sqrt{-49}$.

39. $\sqrt{a^6}\times\sqrt{-a^4}$.

40. $\sqrt{-9y^2}\times\sqrt{-16z^6}$.

41. $\sqrt{-m^2}\times(-\sqrt{-9m^4})$.

42. $i\sqrt{-z^2}$.

43. $i\sqrt{-36}\times\sqrt{25}$.

44. $i\sqrt{-28}\times i\sqrt{-32}$.

45. $i^2\sqrt{-49}\times\sqrt{-121}$.

46. $\sqrt{-(a+b)}\times\sqrt{-(a-b)}$.

47. $\sqrt{x-y}\times\sqrt{y-x}$.

48. $\sqrt{y-5}\times\sqrt{5-y}$.

49. $i\sqrt{-9}\times\sqrt{2-y}$.

50. $(a+\sqrt{-1})(a-\sqrt{-1})$.

51. $(m+bi)(m-bi)$.

52. $(3+5i)(4-7i)$.

53. $(x+2i)(y-bi)$.

54. $(3x-8i)(2y+3i)$.

55. $(2a+3bi)(c-di)$.

56. $(4-i\sqrt{8})(3+i\sqrt{2})$.

57. $(m-3i\sqrt{b})(n+4i\sqrt{c})$.

58. $(\sqrt{5}+i\sqrt{6})(\sqrt{6}-i\sqrt{8})$.

Perform the following divisions:

59. $\frac{a}{\sqrt{-1}}$.

60. $\frac{b}{\sqrt{-b^2}}$.

61. $\frac{d}{\sqrt{-4}}$.

62. $\frac{\sqrt{-9}}{\sqrt{-3}}$.

63. $\frac{\sqrt{-m}}{\sqrt{-n}}$.

64. $\frac{\sqrt{-ax}}{\sqrt{-x}}$.

65. $\frac{x}{\sqrt{-x}}$.

66. $\frac{z}{\sqrt{-z^2}}$.

67. $\frac{m}{i\sqrt{m}}$.

68. $\frac{\sqrt{-x^2}}{\sqrt{-x}}$.

69. $\frac{\sqrt{-8}}{\sqrt{-2}}$.

70. $\frac{\sqrt{-10m^3}}{\sqrt{-5m}}$.

Ex. 50.

Solve: RADICAL EQUATIONS.

1. $2\sqrt{x+5}=\sqrt{28}$.
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7. $6-3\sqrt{x}=4$.
8. $7+2\sqrt[3]{3x}=5$.
9. $11-4\sqrt{5x}=9$.
10. $6+\sqrt{8x}=2$.
11. $\sqrt[3]{2x-3}=-3$.
12. $\sqrt{8y+17}=4$.
13. $\sqrt[3]{3x+7}=3$.
14. $\sqrt[5]{80x-43}=-3$.
15. $14+\sqrt[3]{4x-40}=10$.
16. $\sqrt{2x+8}=\sqrt{5x-2}$.
17. $\sqrt[3]{10y-4}=\sqrt[3]{7y+11}$.
18. $\sqrt[5]{2x-3}=\sqrt[5]{3x+2}$.
19. $2x-8=\sqrt{4x^2-12x+32}$.
20. $\sqrt[3]{3y+2}=\sqrt[6]{9y^2+10y-6}$.
21. $\sqrt{x-1}=\sqrt[8]{(x-1)^2(x+3)^2}$.
22. $\sqrt{(2x-5)}=\sqrt[6]{\frac{1}{8}(2x-5)^4}$.
23. $\sqrt{(3x-4)}=\sqrt[6]{(3x-4)^3(9x-6)}$.
24. $2\sqrt{y-2}=\sqrt[4]{32(y-2)^3}$.
25. $\sqrt[4]{9x^4+6x^3+5x^2-2x}=\sqrt{3x^2+x}$.
26. $\sqrt[6]{8x^3-60x^2+145x-115}=\sqrt{2x-5}$.
27. $\sqrt[3]{(3x^2-5x+2)(2x+4)}=\sqrt[6]{(3x^2-5x+2)^2(4x^2-5x+1)}$.
28. $\sqrt{(5x^5-x^4+2)(5x+3)}=\sqrt[4]{(5x^5-x^4+2)^2(25x^2+25x-6)}$.
29. $3\sqrt{16x+9}=12\sqrt{4x-9}$.
30. $\sqrt{\frac{15}{4}+x}=\frac{5}{2}+\sqrt{x}$.
31. $\sqrt{32+x}=16-\sqrt{x}$.

32. $\sqrt{x}-\sqrt{x-5}=\sqrt{5}$. 34. $\sqrt{x-7}=\sqrt{x+1}-2$.

33. $\sqrt{x-7}=\sqrt{x-14}+1$. 35. $\sqrt{22-x}-2=\sqrt{10-x}$.

36. $\sqrt{x+20}-\sqrt{x-1}-3=0$.

37. $\sqrt{x+15}-7=7-\sqrt{x-13}$.

38. $x=7-\sqrt{x^2-7}$.

39. $\dfrac{2x^2-x+2}{\sqrt{x(2x-1)+2}}=\dfrac{\sqrt{2(x^2+1)-x}}{\sqrt{5x-19}}$.

40. $\dfrac{2(5x^3-22x^2+11x-12)}{3\sqrt{5x^2-2x+3}}=3\sqrt{5x^2-2x+3}$.

41. $\dfrac{24\sqrt[3]{2x+136}+13}{15+4\sqrt[3]{2x+136}}=-5$.

42. $\dfrac{1}{2}-\dfrac{3}{x}=\sqrt{\dfrac{1}{4}-\dfrac{1}{x}\sqrt{9-\dfrac{36}{x}}}$.

43. $\dfrac{\sqrt{x}-3}{\sqrt{x}+3}=\dfrac{\sqrt{x}+1}{\sqrt{x}-2}$.

44. $(2-x)^{\frac{1}{2}}=(4-5x-3x^2)^{\frac{1}{2}}$.

45. $y^{\frac{1}{2}}-y^{-\frac{1}{2}}=(y+1)^{\frac{1}{2}}$.

46. $x^{\frac{1}{2}}+(x-9)^{\frac{1}{2}}=36(x-9)^{-\frac{1}{2}}$.

47. $\dfrac{1+(1-x)^{\frac{1}{2}}}{1-(1-x)^{\frac{1}{2}}}=3$.

48. $[(3+x^2)x^{-3}]^{\frac{1}{2}}=[\frac{1}{8}(9x^{-3}+x^{-2})]^{\frac{1}{2}}$.

49. $\sqrt{2x}+(2x)^{-\frac{1}{2}}(1+x)=2(2x)^{-\frac{1}{2}}$.

50. $\left(\dfrac{y}{5}+4\right)^{\frac{1}{2}}-\left(\dfrac{y}{5}-4\right)^{\frac{1}{2}}=\left(\dfrac{4y-2}{5}\right)^{\frac{1}{2}}$.

51. $(12+x)^{\frac{1}{2}}=x^{\frac{1}{2}}+6$. 52. $(x-3)^{\frac{1}{2}}+x^{\frac{1}{2}}=\dfrac{3}{(x-3)^{\frac{1}{2}}}$.

53. $x^{\frac{1}{2}}-\left(\dfrac{a}{x}\right)^{\frac{1}{2}}=(a+x)^{\frac{1}{2}}$.

54. $\dfrac{2}{x+\sqrt{2-x^2}}+\dfrac{2}{x-\sqrt{2-x^2}}=x.$

55. $\dfrac{\sqrt{1-z}}{2-\sqrt{1+z}}=\dfrac{\sqrt{1+z}}{2+\sqrt{1-z}}.$

56. $\dfrac{1}{1+\sqrt{1-x}}+\dfrac{1}{1-\sqrt{1-x}}=\dfrac{2x}{9}.$

57. $\sqrt{2x-3}=x-3.$

58. $\sqrt{2x-5}+6=x+2.$

59. $\sqrt{2x+1}+2\sqrt{x}=\dfrac{21}{\sqrt{2x+1}}.$

60. $\dfrac{\sqrt{x}}{21-\sqrt{x}}+\dfrac{21-\sqrt{x}}{\sqrt{x}}=2\frac{1}{2}.$

61. $\sqrt{3x^2-6x+\frac{6}{8}}=\sqrt{5x-2x^2-2\frac{2}{9}}.$

62. $\sqrt{(x-1)(3x-6)}=x-2.$

63. $\dfrac{\sqrt{2x-1}}{\sqrt{x^2-9}}=\dfrac{3}{\sqrt{x+11}}.$

64. $\dfrac{\sqrt{2x-3}}{3}+\dfrac{6}{\sqrt{2x-3}}=\dfrac{\sqrt{x-6}}{\sqrt{2x-3}}+\sqrt{2x-3}.$

65. $\sqrt{2(x-1)}=\sqrt[4]{3x^2-x+4}.$

66. $\sqrt[6]{8x^3+150x^2+31x+3}=\sqrt{2x+5}.$

67. $\dfrac{\sqrt{10+x}}{\sqrt{10-x}}=\dfrac{\sqrt{x-2}}{\sqrt{10-x}}+1.$

68. $\sqrt{2(x-3)}=\sqrt[4]{x^2-4x+11}.$

69. $\sqrt[6]{(3x+2)(5x-2)}=\sqrt[3]{2(x+2)}.$

70. $\sqrt{2x+7}=\sqrt{x}+2.$

71. $\sqrt{5x-4}=\sqrt{2x+1}+1.$

72. $(3x-1)^{\frac{1}{2}}-\frac{5}{(x+1)^{\frac{1}{2}}}=\frac{4\sqrt{2}-5}{(x+1)^{\frac{1}{2}}}.$

73. $(x+4)^{\frac{1}{2}}-(5x)^{\frac{1}{2}}=-(x-1)^{\frac{1}{2}}.$

74. $2(3x-\frac{1}{2})^{\frac{1}{2}}+4(3x-\frac{1}{2})^{-\frac{1}{2}}=3(3x+\frac{5}{2})^{\frac{1}{2}}.$

75. $\sqrt{x+17}+\sqrt{x-4}=\frac{7}{4}\sqrt{2x}.$

76. $\sqrt{2x+1}+\sqrt{x-3}=2\sqrt{x}.$

77. $\sqrt{x+4}-\sqrt{x}=\sqrt{x+\frac{8}{2}}.$

78. $\sqrt{2x-\frac{1}{4}}+\sqrt{x+1}=3.$

79. $\sqrt{12+x}=\sqrt{7x+8}-2.$

80. $(x^3+8x^2+16x-1)^{\frac{1}{2}}-x=3.$

81. $(x^4-4x^3+6x^2+15x+9)^{\frac{1}{2}}=(x^2-2x+6)^{\frac{1}{2}}.$

82. $\frac{\sqrt{x^2+x+3}}{\sqrt{2x^2+5x-2}}=\frac{3}{4}.$

83. $\frac{\sqrt{3x^2+x+5}}{\sqrt{4x^2-x+1}}=\frac{3}{2}.$

84. $\frac{(2-3x-x^2)^{\frac{1}{2}}}{(8-5x-2x^2)^{\frac{1}{2}}}=\frac{4}{5}.$

85. $\frac{\sqrt{x^2-6-x}}{7(43+x-x^2)^{\frac{1}{2}}}=\frac{6}{7}.$

86. $\left(\frac{8x^2+12x+1}{4x^2+2x-1}\right)^{-\frac{1}{2}}=\frac{1}{3}.$

87. $\frac{(z^2+2z+1)^{\frac{1}{2}}}{(3z^2+4z+2)^{\frac{1}{2}}}=\frac{2}{3}.$

88. $\left(\frac{\sqrt{x^2-3x+1}}{\sqrt{3x^2+x-50}}\right)^{-1}=\frac{[3(x+5)]^{\frac{1}{2}}}{(x+6)^{\frac{1}{2}}}.$

89. $\frac{4}{x+\sqrt{4-x^2}}+\frac{4}{x-\sqrt{4-x^2}}=\frac{12}{7}.$

90. $\frac{5}{x+\sqrt{5+x^2}}-\frac{5}{x-\sqrt{5+x^2}}=6.$

91. $\dfrac{1}{1-\sqrt{1-x^2}}-\dfrac{1}{1+\sqrt{1-x^2}}=\dfrac{\sqrt{3}}{x^2}.$

92. $\dfrac{5x-1}{\sqrt{5x}+1}=1+\dfrac{\sqrt{5x}-1}{2}.$

93. $\dfrac{4x-1}{\sqrt{2x+1}}+3\sqrt{2x+1}=7\sqrt{x}.$

94. $\dfrac{\sqrt{3x^2+4}-\sqrt{2x^2+1}}{\sqrt{3x^2+4}+\sqrt{2x^2+1}}=\dfrac{1}{7}.$

95. $\dfrac{\sqrt{27x^2+4}+\sqrt{9x^2+5}}{\sqrt{27x^2+4}-\sqrt{9x^2+5}}=7.$

96. $\dfrac{\sqrt{5x-4}+\sqrt{5-x}}{\sqrt{5x-4}-\sqrt{5-x}}=\dfrac{\sqrt{4x}+1}{\sqrt{4x}-1}$

97. $\sqrt{ax^2-bx-b}=\sqrt{-ax}.$

98. $\dfrac{\sqrt{x+a}}{\sqrt{3x^2-5ax+\dfrac{4a^2}{3}}+1}=\dfrac{\sqrt{x^2+3ax+2a^2}}{\sqrt{x+2a}}.$

99. $\dfrac{\sqrt{ax+b}+\sqrt{ax}}{\sqrt{ax+b}-\sqrt{ax}}=\dfrac{1+\sqrt{ax-b}}{1-\sqrt{ax-b}}.$

100. $\dfrac{4x}{a-b}=\dfrac{1}{\sqrt{a}-x}.$

101. $\sqrt{a+x}-\sqrt{a-x}=\sqrt{a}.$

102. $\sqrt{6x^2+5a^2+2ax}=a+3x.$

103. $\sqrt{10ax-x^2+25a^2}=x+3a.$

104. $\sqrt{\dfrac{(3x+2b)x+\frac{2}{3}(\frac{1}{2}b^2-2ab)}{3}}=\dfrac{a-b}{3}.$

105. $\sqrt{x}+\sqrt{2a-x}=\frac{a}{\sqrt{x}}$.

106. $\frac{2x(\sqrt{a+x}+\sqrt{a-x})}{\sqrt{a+x}-\sqrt{a-x}}=x+2a$.

107. $\frac{\sqrt{x}+\sqrt{b}}{\sqrt{x}-\sqrt{b}}=\frac{\sqrt{a-x}+\sqrt{b-x}}{\sqrt{a-x}-\sqrt{b-x}}$.

108. $\sqrt{x-m}-\sqrt{x-n}=\sqrt{n-m}$.

109. $b\sqrt{b^2-x^2}-\frac{b(x^2-7b^2)}{\sqrt{b^2-x^2}}=0$.

110. $\sqrt{x^2+x+1}=a-\sqrt{x^2-x+1}$.

111. $\sqrt{x}+\sqrt{a-\sqrt{ax+x^2}}=\sqrt{a}$.

112. $\sqrt{a-x}+\sqrt{-(a^2+ax)}=\frac{a}{\sqrt{a-x}}$.

113. $\sqrt{\frac{x}{b}}+\sqrt{b^3x+4ab}=\sqrt{b^3x+\frac{x}{b}+4ab+\frac{3a}{b}}$.

114. $\frac{\sqrt{4a^2-x^2}+p\sqrt{4a-x}}{\sqrt{4a^2-x^2}-p\sqrt{4a-x}}=\frac{\sqrt{a}+p}{\sqrt{a}-p}$.

115. $\frac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{x+b}+\sqrt{x-b}}=\frac{\sqrt{a+x}-\sqrt{a-x}}{\sqrt{x+b}-\sqrt{x-b}}$.

116. $\frac{\sqrt{a-x}}{x}-\frac{\sqrt{a-x}}{a}=\sqrt{x}$.

117. $\frac{\sqrt{a-x}+\sqrt{x-b}}{\sqrt{a-x}-\sqrt{x-b}}=\sqrt{\frac{a-x}{x-b}}$.

118. $\frac{\sqrt{x+2a}-\sqrt{x-2a}}{\sqrt{x-2a}+\sqrt{x+2a}}=\frac{x}{2a}$.

119. $\dfrac{a-x}{\sqrt{a-x}}+\dfrac{x-b}{\sqrt{x-b}}=\sqrt{a-b}.$

120. $\sqrt{\dfrac{a-x}{b+x}}-\sqrt{\dfrac{b+x}{a-x}}=c.$

121. $\sqrt{\dfrac{a-x}{b-x}}+\sqrt{\dfrac{b-x}{a-x}}=c.$

122. $\sqrt{\dfrac{a-x}{b-x}}-\sqrt{\dfrac{b-x}{a-x}}=c.$

123. $\dfrac{x^2}{8a}+\dfrac{2x}{3}=\sqrt{\dfrac{x^3}{3a}+\dfrac{x^2}{4}}-\dfrac{a}{2}.$

Ex. 51.

SOLUTION OF PROBLEMS BY LOGARITHMS.

Find by logarithms the amounts at compound interest of:

1. \$2640 at 5 %, for 8 years.
2. \$7800 at 4½ %, for 13 years.
3. \$2315 at 2½ %, for 30 years.
4. \$5240 at 3¼ %, for 60 years.
5. \$745.20 at 5¼ %, for 12 years.
6. \$500 at 5 %, for 200 years.
7. \$1 at 6 %, for 150 years.
8. \$.20 at 4¼ %, for 70 years.
9. \$.05 at 3¾ %, for 400 years.
10. \$3000 at 3 %, for 21 years and 134 days.
11. \$800 at 1½ %, for 91 years, 219 days.
12. \$415 at 4½ %, for 8 years, 7½ months.

13. What will $600 become at 5%, for 18 years, the interest being compounded every six months?

14. What is $1600 worth at the end of 10 years at 4%, the interest being compounded: (i.) every year? (ii.) every six months? (iii.) every three months? (iv.) every month?

Find what capital must be put at compound interest to produce:

15. $13,823.13 in 13 years, at 4½%.

16. $1927.65 in 10 years, at 3½%.

17. $6411.86 in 20 years, at 2%.

18. $1600 in 15 years, at 3½%.

19. $2000 in 18 years, at 5%.

20. What is the actual value of a sum of $2400, payable in 9 years, and discounted at compound interest at 4½%?

Find during what time it is necessary to place, at compound interest:

21. $620 at 2½%, to produce $755.41.

22. $2412 at 4½%, to produce $3914.32.

23. $1200 at 4%, to produce $1600.

24. $1400 at 5%, to produce $20,000.

25. In what time will a sum, placed at compound interest, double itself: (i.) at 3%? (ii.) at 3½%? (iii.) at 4%? (iv.) at 5%?

Find at what rate it is necessary to lend, at compound interest:

26. $1800, to produce $12,791.68 in 50 years.

27. $918, to produce $1757.92 in 14 years.

28. $2500, to produce $3715.24 in 9 years.

29. $856.40, to produce $1546.75 in 20 years.

30. $1, to produce $18.42 in 50 years.

31. At the birth of a child, there is placed in a bank, which pays compound interest at $4\frac{1}{2}\%$, a sum of $500. How much money will he receive at the age of 21 years?

32. A man, who has to pay a sum of $4520 on June 30, 1880, pays his debt March 31, 1877, getting a compound discount of $4\frac{1}{2}\%$. How much does he pay?

33. A usurer pays $1000 for a note of $1800, payable in three years. What rate of compound interest does the usurer receive?

34. If a bank pays to its depositors 4%, and lends the money at $4\frac{3}{4}\%$, how much will the bank gain in 10 years upon $1000, reckoning compound interest in both cases?

35. The population of a certain city has increased in 24 years from 32,500 to 66,066 people. How much per cent has it increased each year?

36. A cask contains 140 gallons of wine. A person draws out 4 gallons, which he replaces by water. The same operation is repeated 24 times, taking care to mix each time the water equally with the wine. How much wine remains in the cask?

37. A forest contains to-day 120,000^{cbm} of wood. How much will it contain in 15 years, if the annual increase is $3\frac{1}{4}\%$?

38. When Jacob arrived in Egypt, his family was composed of 70 persons; 430 years after, when the Israelites went out, they numbered 660,000. How much per cent was the increase each year, if from 1000 persons 25 died annually?

39. What is the principal which, at the end of 8 years, at 4 %, is worth as much as $2700, at 3 %, in 12 years?

40. Two sums of money, of which the second is worth $1420 more than the first, are placed at compound interest, the first at 4 %, the second at 5 %. At the end of 16 years they are worth, together, $211,084. What are these sums of money?

41. Two sums of money, one being $393 greater than the other, are placed at compound interest, the smaller at 5¼ %, the greater at 3¼ %. What are these sums of money, knowing that, at the end of 40 years, the smaller is double the other?

42. A man lends at the same time at compound interest $38,000 at 4½ %, and $99,398 at 3½ %. In what time will the two sums be equal?

43. A house is for sale. A offers $30,000 cash; B, $35,000, payable in 3 years; C, $33,000, in three equal sums, payable at the commencement of each year, from to-day. Which is the most advantageous offer, and how much?

If, at the end of each year, commencing in a year, there is placed during n years a sum s, at compound interest, the value S of all these annuities and of their interests is given by the formula

$$S = \frac{s}{r}[(1+r)^n - 1].$$

If to-day there is placed a capital C, and each year, to commence in a year, s dollars are added to this capital, at the end of n years the value of S is given by the formula

$$S = C(1+r)^n + \frac{s}{r}[(1+r)^n - 1].$$

It follows that, to pay in n years, by an annuity s, commencing in one year, a capital C borrowed to-day, we have

$$C(1+r)^n = \frac{s}{r}[(1+r)^n - 1].$$

The actual value A, which must be given to-day to buy up an annuity of s dollars, is given by the following formulas. If the annuity commences:

1. In one year to be perpetual,

$$A = \frac{s}{r}.$$

2. In $(m+1)$ years to be perpetual,

$$A = \frac{s}{r(1+r)^m}.$$

3. In one year to cease in m years,

$$A = \frac{s}{r} - \frac{s}{r(1+r)^m} = \frac{s[(1+r)^m - 1]}{r(1+r)^m}.$$

4. In $(m+1)$ years and ceases p years later,

$$A = \frac{s}{r(1+r)^m} - \frac{s}{r(1+r)^{m+p}} = \frac{s[(1+r)^p - 1]}{r(1+r)^{m+p}}.$$

44. A smoker, who commenced to smoke at the age of 17, spends \$1.80 per week. If, renouncing this habit, he had, at the end of his eighteenth year, placed in a bank at 4 % the sum that he had spent for tobacco, and, if he had done the same at the end of each year, how much would he have possessed at the age of 60 years?

45. During how many years must a sum of \$250 be placed at $4\frac{1}{2}$ %, in order that the principal and interest may amount to \$8195.76?

46. An engineer who, each year, has taken $500 from his salary in order to place it at compound interest at 5 % a year, finds himself, at the age of 54 years, a possessor of $34,880.37. At what age did he commence his deposits?

47. What is the total value, after the twenty-sixth deposit, of an annuity of $1400, deposited half June 30, and half December 31, the interest at 4 % being compounded every six months?

48. A man has placed in a bank, December 31, 1874, a sum of $2800, at compound interest at 4 %. At the end of December, in each of the following years, he adds to the capital $450. How much will he possess at the end of December, 1887?

49. In a city of 800,000 people, where the immigration brings each year 2000 persons, the population, both the old and the new, increases each year 2 %. What will be the population of the city in 15 years?

50. A father died, leaving a fortune of $80,000, which produces 4½ %. In order to bring up the children, an annual sum of $5000 must be spent for 8 years. What will remain of the property at the end of this time?

51. A stockholder, who possesses a capital of $120,000, spends each year $6800 for his support. The interest being at 4½ %, how much will he have left in 20 years?

52. A forest contains 80,000 cords of wood, and increases 3 % per year. How many cords of wood can be cut each year, so that in 30 years the forest will contain 100,000 cords?

53. I borrow \$18,000 at 4½ %, and pay each year 10 % of this sum. In how many years will my debt be paid?

54. A man inherits a fortune of \$120,000, producing interest at 5½ %. He spends, besides what he earns, \$9500 per year. In how many years will he have spent his fortune?

55. A man borrows, at 5 %, a sum of \$50,000, which he places in business, producing \$6000 per year. Using all the proceeds for the payment of the interest and the principal, in how many years will the debt be paid?

56. How much must be given to-day to redeem a perpetual annuity of \$500, which is to commence in a year, if the interest is at 4 %?

57. What is the actual value of a perpetual annuity of \$1200, which is to commence in 12 years, if the interest is at 4½ %?

58. A sum of \$18,324.46, due to-day, is to be converted into a perpetual annuity of \$2000. In how many years ought the first payment to be made, if the interest is at 5 %?

59. An annual rent of \$2800, which must be paid 14 years more, is changed to another annual rent payable during 20 years. To how much will the last be raised, the interest being reckoned at 4½ %?

60. How much must be given to-day to redeem, at 5 %, an annuity of \$9500, which commences in 10 years, and lasts 40 years?

61. In order to redeem, at 6 %, an annuity which commences in 25 years, and lasts 50 years, \$19,464.20 are given. How much is the annuity?

62. A man pays \$10,200 to redeem a perpetual annuity of \$459, which commences in 1 year. At what rate is the interest reckoned?

Ex. 52.

RATIO AND PROPORTION.

1. An express-train goes from A to B in $2\frac{1}{4}$ hours; a coach accomplishes the same distance in 9 hours. What is the ratio of the two velocities?

2. A hall is 39^m long and 26^m wide. Express the ratio of the length to the width.

3. The diameter of a column of the Tuscan order is 1.17^m and its height is 8.19^m. Express the ratio of the diameter to the height.

4. The diameter of a column of the Corinthian order is $2\frac{7}{12}$ feet and its height is $25\frac{5}{6}$ feet. What is the ratio of the diameter to the height?

5. The height of a column of the Ionian order is $24\frac{3}{4}$ feet and its diameter is $2\frac{3}{4}$ feet. Find the ratio of the height to the diameter.

6. The diameter of a column being $1\frac{5}{6}$ feet, the height in the Tuscan order will be $12\frac{5}{6}$ feet, in the Dorian order $14\frac{2}{3}$ feet, in the Ionian order $16\frac{1}{2}$ feet, and in the Corinthian order $18\frac{1}{3}$ feet. What are the ratios of these heights, the diameters being the same?

7. If two workmen, A and B, are paid in proportion to their activity, in what ratio will be their salaries if to do the same work A employs 18 days and B 22 days?

8. A garrison has food for 260 days; it receives orders to hold out 60 days more with the same provisions. What is the ratio of the new ration to the old?

9. Two pedestrians walk 75 yards and 90 yards per minute. In what ratio will be the time that they each use to accomplish the same distance?

10. The lengths of two rectangular fields are to each other as 2 : 3; the widths are in the ratio of 5 : 6. What is the ratio of their surfaces?

11. The volumes of two stones are to each other as 3 : 4; the weights of equal volumes are as 2 : 5. What is the ratio of the weight of the first stone to that of the second?

12. The surfaces of two circles are as the squares of their radii. Find the ratio of the surfaces of two circles, the first having a radius of 18 feet, and the other a radius of 24 feet.

13. The volumes of two spheres are to each other as the cubes of their radii. In what ratio are the volumes of two spheres having radii of 8 feet and 12 feet?

14. Of two moving bodies one accomplishes m feet in t seconds, the other n feet in p seconds. What is the ratio of their velocities? What is the ratio if $m = 35$, $t = 9$, $n = 25$, $p = 6$?

Form a proportion with the numbers:

15. 75, 18, 27, 50.

16. 969, 663, 285, and 195.

Find x in:

17. $27 : 90 = 45 : x$.

18. $10 : 56 = x : 21$.

19. $3.15 : x = 6.75 : 20$.

20. $x : 3\frac{3}{4} = 4\frac{1}{2} : 11\frac{1}{4}$.

21. $3\frac{3}{4} : 5\frac{2}{5} = 7\frac{1}{2} : x.$

22. $5\frac{2}{5} : 3\frac{3}{4} = x : 7\frac{7}{8}.$

23. $\frac{4}{7} = \frac{8}{x}.$

24. $\frac{3}{7} = \frac{x}{42}.$

25. $\frac{24\frac{2}{3}}{15\frac{1}{2}} = \frac{2x}{7\frac{3}{4}}.$

26. $\frac{x}{1.5} = \frac{1.5}{1.8}.$

27. $\frac{3}{4}mn : \frac{5}{6}mp = x : 3\frac{1}{5}p.$

28. $1\frac{9}{8}h^2q : \frac{5}{6}ht = \frac{4}{5}hq^2 : x.$

29. $\frac{221\,a^2b}{51\,pq} = \frac{26\,ab^2}{x}.$

30. $\frac{17(m+n)}{x} = \frac{51(m^2-n^2)}{13(m-n)}.$

31. $\frac{3a}{5b} : \frac{12a}{7c} = \frac{14c}{15b} : x.$

32. $\frac{15(a+b)}{14(a-b)} : \frac{10(a+b)^2}{21(a-b)^2} = \frac{a-b}{a+b} : x.$

33. $\left(\frac{m^3-n^3}{m-n} - mn\right) : \left(\frac{m^3+n^3}{m+n} + mn\right) = 1 : x.$

34. $\left(\frac{p^3-q^3}{p-q} - pq\right) : (p+q)^2 = \left(\frac{p^3+q^3}{p+q} + pq\right) : x.$

35. $\frac{1}{m-n} : \frac{m^2-mn+n^2}{m^3-3mn(m-n)-n^3} = \frac{m^2-n^2}{m^3+n^3} : x.$

36. $\frac{a^2-b^2}{ac+c^2} : \frac{a+b}{a^2-c^2} = x : a + \frac{ac}{a-c}.$

37. $\frac{b(1+b)}{1-a^2} : \frac{1-b^2}{a(1+a)} = \left(b + \frac{ab}{1-a}\right) : x.$

38. $\frac{m^2-4m+4}{m^2-4m+3} : x = \frac{m^2-5m+6}{m^2-2m+1} : \frac{m^2-6m+9}{m^2-3m+2}.$

Find a fourth proportional to:

39. 5, 6, 10.

40. 8, 7, 5.

41. $3\frac{3}{4}$, $7\frac{1}{2}$, $22\frac{2}{3}$.

42. m, n, p.

43. $\frac{1}{a}$, $\frac{1}{b}$, $\frac{1}{c}$.

44. $\frac{m-n}{m+n}$, $\frac{(m-n)^2}{(m+n)^2}$, $\frac{m}{n}$.

45. $\frac{1+n}{1-m^2}$, $\frac{1-n^2}{m+m^2}$, $1+\frac{m}{1-m}$. 46. $\frac{a^2-b^2}{a^2+b^2}$, $1+\frac{a}{b}$, $1-\frac{b}{a}$.

Find a third proportional to:

47. 9, 6.

48. 16, 12.

49. 9, 15.

50. 8, 9.

51. m, n.

52. $\frac{p^2-q^2}{r}$, $\frac{p-q}{r}$.

53. $\frac{p}{(p+m)^2}$, $\frac{2p^2}{m^3+p^3}$.

54. $\frac{m^2}{l^2-m^2}$, $\frac{lm-m^2}{(l+m)^2}$.

55. $\frac{36a^2b^2}{(a^2-b^2)^2}$, $\frac{4(a^2-ab)}{b(a+b)^2}$.

Find a mean proportional between:

56. 27 and 3.

57. 3 and 12.

58. 5 and 20.

59. 2 and $12\frac{1}{2}$.

60. 3 and $16\frac{1}{3}$.

61. $\frac{mn}{p}$ and $\frac{mp}{n}$.

62. $\frac{2p}{q}$ and $\frac{q^3}{8p^3}$.

63. $\frac{m-5}{m^2+10m+25}$ and $\frac{(m+5)^3}{m^2-25}$.

64. $\frac{c+d}{f-g}$ and $\frac{(f^2-g^2)(c+d)}{f+g}$.

Ex. 53.

Arithmetical Progression.

$$l=a+(n-1)d;\ s=\frac{n}{2}(a+l)=\frac{n}{2}[2a+(n-1)d].$$

Find l and s in:

1. 7, 11, 15 ..... (13 terms).

2. 5, 8, 11 ..... (12 t.).

3. $\frac{1}{2}$, $6\frac{1}{2}$, $12\frac{1}{2}$ ..... (30 t.).

4. 1, 1.1, 1.2 ..... (200 t.).

5. $\frac{4}{7}$, $\frac{2}{7}$, $\frac{1}{7}$ ..... (15 t.).

6. 63, 58, 53 ..... (8 t.).

7. $3n$, $5n$, $7n$ ..... (36 t.).

8. $2m$, $2m+4n$, $2m+8n$ ..... (14 t.).

9. 1, $1+3m$, $1+6m$ ..... (11 t.).

Find d and s in:

10. 2 ..... 134, (13 t.).

11. 0 ..... 200, (51 t.).

12. $3\frac{1}{4}$ ..... 64, (82 t.).

13. 169 ..... 8, (24 t.).

14. 7 ..... -3.5, (36 t.).

15. $5a^3$ ..... $145a^3$, (21 t.).

16. $2b^3$ ..... $(2b^3+17a^3)$, (35 t.)

Insert between:

17. 13 and 76, eight means.

18. 1 and 2, fifteen means.

19. $\frac{1}{2}$ and $\frac{3}{4}$, eight means.

20. 47 and 2, eight means.

Find a and s in:

21. ..... 142, 149; $n=22$.

22. ..... 221, 242; $n=12$.

23. ..... $26\frac{1}{3}$, $27\frac{5}{6}$; $n=19$.

24. ..... -20, -22; $n=40$.

25. ..... 5, 0; $n=14$.

Find n and s in:

26. 17 ..... 350; $d=9$.

27. 34 ..... 10; $d=-2$.

28. $116\frac{1}{4}$ ..... 0; $d=-\frac{5}{4}$.

Find d and n in:

29. $1\frac{1}{2}$ ..... 54; $s=999$.

30. 5 ..... 161; $s=3320$.

31. 2 ..... 87; $s=801$.

32. -13 ..... 27; $s=77$.

Find d and l when:

33. $a=10$; $n=14$; $s=1050$.

34. $a=1$; $n=20$; $s=305$.

35. $a=0$; $n=30$; $s=290x$.

36. $a=16$; $n=9$; $s=0$.

Find a and d when :

37. $l = 21$; $n = 7$; $s = 105$.

38. $l = 148$; $n = 27$; $s = 2241$.

39. $l = 11\frac{2}{3}$; $n = 37$; $s = 209\frac{2}{3}$.

40. $l = 105$; $n = 16$; $s = 840$.

Find a and l when :

41. $n = 21$; $s = 1197$; $d = 4$.

42. $n = 12$; $s = 594$; $d = 7$.

43. $n = 25$; $s = -75$; $d = \frac{1}{2}$.

44. $n = 33$; $s = -33$; $d = -\frac{3}{4}$.

Find n and l when :

45. $s = 1095$; $a = 38$; $d = 5$.

46. $s = 624$; $a = 9$; $d = 4$.

47. $s = 2551.3$; $a = 32$; $d = 0.3$.

48. $s = 1782$; $a = 18$; $d = 6$.

Find a and n when :

49. $s = 623$; $d = 5$; $l = 77$.

50. $s = 1008$; $d = 4$; $l = 88$.

51. $s = 281\frac{1}{7}$; $d = \frac{3}{7}$; $l = 15\frac{3}{7}$.

52. $s = 2640$; $d = -20$; $l = 15$.

53. When a train arrives at the top of a long slope, the last car is detached and begins to descend, passing over 50^{cm} in the first second, three times 50^{cm} in the second, five times 50^{cm} in the third, etc. At the end of 12 minutes it reaches the bottom of the slope. What was its velocity in the last second?

54. A man wishes to have his horse shod. The blacksmith asks him $2 a shoe, or 1 cent for the first nail, 3 for the second, 5 for the third, etc. Each shoe has 8 nails. Ought the man to accept the second proposition?

55. There are placed in a straight line upon a lawn 50 eggs 3 feet distant from each other. A person is required to pick them up one by one and carry them to a basket 3 feet from the first egg, while a runner, starting from the basket, touches a goal and returns. At what distance ought the goal to be placed that both men may have the same distance to pass over?

56. Starting from a box, there are placed upon a straight line 40 stones, at the distances 1 foot, 3 feet, 5 feet, etc. A man placed at the box is required to take them and carry them back one by one. What is the total distance that he has to accomplish?

57. A stone which falls freely passes over respectively in the first three seconds of its fall 4.9^m, 3×4.9^m, and 5×4.9^m. In the famous balloon ascension which was made in 1875, the balloon rose to 8643.6^m. What velocity would a stone have reached falling from the balloon to the ground?

58. A body, which moves vertically, moves with a velocity which decreases in the same manner as the velocity of a falling body increases; in the last second the rising body passes over 4.9^m; in the second before the last, 3×4.9^m; in the preceding second, 5×4.9^m. Supposing that a ball is shot vertically from a field-piece with a velocity of 445.9^m per second, find (i.) during what time it will rise; (ii.) what height it will reach?

59. How much time will it take a stone, falling from the top of the dome of the Pantheon at Paris, to reach the ground? The highest point of the cupola is 83^m.

60. The product of five numbers in arithmetical progression is 379,440, and their sum is 85. Find them.

61. The sum of three numbers in arithmetical progression is 33, and the sum of their squares is 461. What are the numbers?

62. The sum of five terms in arithmetical progression is 50, and the sum of their squares 590. Find the terms.

63. The sum of 11 numbers in arithmetical progression is 77, and the sum of their squares 1529. What are these numbers?

Ex. 54.

Arithmetical Progression of a Higher Order.

If we represent by S_2 the sum of the squares, and by S_3 the sum of cubes of the natural numbers from 1 to n inclusive, we have:

$$S_2 = \frac{n(n+1)(2n+1)}{1\times 2\times 3}. \qquad S_3 = \left(\frac{n(n+1)}{1\times 2}\right)^2.$$

If we represent by N the number of balls contained in a pile, and by n the number of balls upon each side of the lower tier, we have:

Pile with a square base,

$$N = \frac{n(n+1)(2n+1)}{1\times 2\times 3}.$$

Pile with a triangular base,

$$N = \frac{n(n+1)(n+2)}{1\times 2\times 3}.$$

Pile with a rectangular base,

$$N = \frac{n(n+1)(3m-n+1)}{1\times 2\times 3}.$$

In this last formula, m is the number of balls upon the large side of the base, and n the number upon the small side.

What is the sum of the squares of the natural numbers between the following limits, inclusively:

1. From 1 to 13?
2. From 1 to 40?
3. From 1 to 53?
4. From 1 to 100?
5. From 1 to 125?
6. From 1 to 315?
7. From 1 to 1000?
8. From 50 to 200?

What is the sum of the cubes of the natural numbers between the following limits, inclusively:

9. From 1 to 15?
10. From 1 to 39?
11. From 15 to 29?
12. From 20 to 50?
13. From 32 to 90?
14. From 7 to 35?

What is the number of balls contained in a complete pile with a square base, having upon each side of its base:

15. 6 balls?
16. 9 balls?
17. 7 balls?
18. 17 balls?
19. 21 balls?
20. 27 balls?

What is the number of balls contained in a truncated pile with a square base, having:

21. Lower base, 11 balls; upper base, 5 balls?
22. Lower base, 9 balls; upper base, 4 balls?
23. Lower base, 14 balls; upper base, 6 balls?
24. Lower base, 17 balls; upper base, 9 balls?
25. Lower base, 18 balls; upper base, 5 balls?
26. Lower base, 16 balls; upper base, 6 balls?

How many balls does a complete triangular pile contain, having upon each side:

27. 6 balls?
28. 12 balls?
29. 17 balls?
30. 10 balls?
31. 19 balls?
32. 25 balls?

How many balls does a truncated triangular pile contain, having:

33. Lower base, 14 balls; upper base, 6 balls?
34. Lower base, 17 balls; upper base, 13 balls?
35. Lower base, 24 balls; upper base, 17 balls?
36. Lower base, 20 balls; upper base, 14 balls?

How many balls does a rectangular pile contain, having upon the two sides of the lower base:

37. 11 and 5 balls?
38. 9 and 4 balls?
39. 21 and 13 balls?
40. 24 and 17 balls?
41. 31 and 19 balls?
42. 27 and 23 balls?

Find the number of balls which form a rectangular truncated pile, having:

43. Lower base, 19 and 14 balls; upper base, 15 and 10 balls.
44. Lower base, 22 and 16 balls; upper base, 16 and 10 balls.
45. Lower base, 18 and 11 balls; upper base, 11 and 4 balls.
46. Lower base, 28 and 19 balls; upper base, 19 and 10 balls.

Ex. 55.

Geometrical Progression.

$$l = ar^{n-1}, \quad S = \frac{lr - a}{r - 1} = \frac{a(r^n - 1)}{r - 1},$$

or, for decreasing progressions,

$$S = \frac{a(1 - r^n)}{1 - r}.$$

Find l and s:

1. 4, 8, 16 ..... (7 terms).
2. 2, 6, 18 ..... (9 t.).
3. 1, 4, 16 ..... (7 t.).
4. 9, 3, 1 ..... (11 t.).

5. 13, 1.3, 0.13 ..... (6 t.).
6. 56, 28, 14 ..... (12 t.).
7. 1, $\frac{1}{2}$, $\frac{1}{4}$ ..... (10 t.).
8. 1, 1.5, 2.25 ..... (6 t.).
9. 1, $\frac{1}{2}$, $\frac{1}{4}$ .....
10. 8, 2, $\frac{1}{2}$ .....
11. 2, $\frac{2}{7}$, $\frac{2}{49}$ .....
12. 5, 3, $\frac{9}{5}$ .....
13. 1, $\frac{1}{4}$, $\frac{1}{16}$ .....
14. m^3, m^3n, m^3n^2, (7 t.).
15. a, $a(1+x)$, $a(1+x)^2$ ..... (8 t.).
16. a, $\frac{a}{a^2-1}$, $\frac{a}{(a^2-1)^2}$ ..... (10 t.).
17. b, $\frac{b}{1-m}$, $\frac{b}{(1-m)^2}$ ..... (9 t.).
18. $b(1+x)^{n-1}$, $b(1+x)^{n-2}$, $b(1+x)^{n-3}$ ..... (n t.).

Find r and s when :

19. 8 ..... 10,368, (5 t.).
20. 2 ..... 31,250, (7 t.).
21. 36 ..... $\frac{4}{81}$, (7 t.).
22. 3 ..... 49,152, (8 t.).
23. 48 ..... $\frac{3}{64}$, (6 t.).
24. 7 ..... 3584, (10 t.).

Between the following terms insert the number of means indicated :

25. 14 and 686, (1 mean).
26. 47 and 1269, (2 m.).
27. 31 and 496, (3 m.).
28. 13 and 101,088, (4 m.).
29. 81 and $\frac{256}{81}$, (7 m.).
30. 128 and 31,250, (5 m.).
31. $\frac{5}{64}$ and 640, (12 m.).
32. $\frac{2}{243}$ and 1458, (10 m.).
33. 6250 and $\frac{9}{81250}$, (9 m.).

Find a and s when :

34. $l = 128$; $r = 2$; $n = 7$.
35. $l = 78{,}125$; $r = 5$; $n = 8$.
36. $l = 29{,}296{,}875$; $r = 5$; $n = 11$.

37. $l = 14\frac{2150}{2187}$; $r = \frac{4}{3}$; $n = 8$.

38. $l = \frac{15809}{262144}$; $r = \frac{3}{8}$; $n = 8$.

39. $l = \frac{2}{27}$; $r = \frac{1}{3}$; $n = 5$.

Find s and n when:

40. $a = 9$; $l = 2304$; $r = 2$.

41. $a = 2$; $l = 64$; $r = 2$.

42. $a = 3$; $l = 192\sqrt{2}$; $r = \sqrt{2}$.

43. $a = 54$; $l = \frac{2}{243}$; $r = \frac{1}{3}$.

44. $a = 9$; $l = \frac{32}{27}$; $r = \frac{2}{3}$.

Find r and n when:

45. $a = 2$; $l = 1458$; $S = 2186$.

47. $a = 10$; $l = \frac{5}{16}$; $S = 19\frac{11}{16}$.

46. $a = 1$; $l = 2401$; $S = 2801$.

48. $a = 3125$; $l = 5$; $S = 3905$.

Find the two unknown quantities in the following examples:

49. $a = 3$; $r = 3$; $S = 29{,}523$.

51. $r = 2$; $n = 7$; $S = 635$.

53. $l = 1296$; $r = 6$; $S = 1555$.

50. $a = 8$; $r = 2$; $S = 4088$.

52. $r = 8$; $n = 6$; $S = 149{,}796$.

54. $l = 32{,}768$; $r = 4$; $S = 43{,}690$.

55. The first term of a geometrical progression is 18, the number of terms is 3, and the sum of the terms is 1026. Find the last term and the ratio.

56. A cask contains 240 gallons of wine. The first day of January, half is drawn out; the following day, half of the rest; the third day, half of what then remains; and so on. How much will be drawn off on January 31?

57. What is the value of the decimal fraction 0.7272 .....?

58. The sum of the two extreme terms of a geometrical progression of four terms is 195; the sum of the two mean terms is 60. What is the first term, and what is the ratio?

59. What is the sum of the terms of the progression 1, -3, 9, -27, to the twentieth term, inclusively?

60. What is the sum of the first twelve terms of the progression, $-2, +4, -8, +16$ .....?

61. What is the sum of the terms to infinity of the progression $9, -3, +1, -\frac{1}{3}$ .....?

Ex. 56.

PERMUTATIONS AND COMBINATIONS.

The permutations of m letters are the results obtained by writing, in every possible order, these m letters, one after another, so that each letter may occur in each permutation once, and once only.

If P_m is the number of permutations of m letters, we have

$$P_m = 1 \times 2 \times 3 \times 4 \ldots\ldots m = \lfloor\underline{m}.$$

If in m objects there are p objects of one kind, q of a second kind, r of a third kind, the number of possible permutations is given by

$$P_m = \frac{1 \times 2 \times 3 \ldots\ldots m}{(1 \times 2 \times 3 \ldots\ldots p)(1 \times 2 \times 3 \ldots\ldots q)(1 \times 2 \times 3 \ldots\ldots r)} = \frac{\lfloor\underline{m}}{\lfloor\underline{p}\,\lfloor\underline{q}\,\lfloor\underline{r}}.$$

If P_n is the number of permutations of m things, taken n at a time ($n < m$), then

$$P_n = m(m-1)(m-2)\ldots\ldots(m-n+1).$$

1. In how many different ways can the 26 letters of the alphabet be placed, one after another?

2. In how many ways can 14 persons be placed at table?

3. In how many ways can the 30 pupils of a class be placed?

4. In how many ways can the 8 bells of a chime be struck successively?

5. In how many ways can 25 men be placed in a rank?

6. It is required to place 20 vases of flowers in a half circle. In how many different ways can they be arranged?

7. How many numbers can be formed with the figures 1, 2, 3, 4, 5, 6, 7, 8, 9, each figure occurring in each number once, and once only?

8. In the same case, how many different numbers can be formed with the figures 1, 2 ..... 9, 0? The numbers beginning with zero not to be counted.

9. In how many different ways can be placed, one after another, 24 colored eggs, of which 4 are yellow, 8 blue, 7 brown, and the others red?

10. In a library are 30 volumes, of which 10 are bound in white, 9 in red, 6 in green, and the 5 others in brown. In how many ways can they be placed as to their colors?

How many permutations can be made with the following products:

11. a^2bc. 12. $a^3b^7c^2$. 13. $a^7x^5cd^4$. 14. $a^5x^5y^8z^2$.

15. How many words can be made with 9 letters, 3 letters at the beginning remaining inseparable, and keeping the same order?

16. What will be the answer to the preceding problem if the three inseparable letters can be arranged in any order?

17. With 7 letters, among which are several a's, 210 different words can be formed. How many a's are there?

18. How many arrangements can be formed with 12 objects taken 2 at a time?

19. How many arrangements can be formed with 30 objects taken 8 at a time?

20. A girl has two rings of different form. In how many ways can she place them upon her fingers, either on the left hand or on the right, putting one ring on a finger? (The thumb does not count.)

21. How many arrangements can be formed with 15 objects taken 6 at a time?

22. How many arrangements can be formed with 26 letters, if 4 are taken at a time? if 6 are taken at a time?

23. How many numbers of three figures can be formed with the numbers 1 to 9 inclusive?

24. With the figures 1, 2, 3, 4, 5, 6, 7, 8, 9, 0, how many numbers can be formed: (i.) of 3 figures? (ii.) of 4 figures? The numbers beginning with zero not to be counted.

25. How many words can be formed with 20 consonants and 6 vowels, each word containing 3 consonants and 2 vowels, and the latter occupying only the second and fourth places?

26. What would be the answer to the preceding problem if the vowels could occupy any place whatever?

27. How many words of 3 letters can be made with 20 consonants and 6 vowels, taking two consonants and a vowel for each word, if the vowels can represent in all, by means of accents, 10 different sounds?

28. In how many ways can the 10 bells of a chime be struck: (i.) 4 at a time? (ii.) 5 at a time?

29. The number of arrangements of n objects taken 3 at a time is to the number of arrangements of the same objects, taken 4 at a time, as 1 is to 20. Find n.

30. A number of m objects is to the number of its arrangements, taken 3 at a time, as 1 is to 240. Find m.

31. How many numbers of 4 figures can be formed: (i.) with the figures 1, 2, 3, 4; (ii.) with the figures from 1 to 9 inclusive, if in both cases the figures can be repeated?

32. What is the number of the arrangements, with repetitions, of twelve objects taken 5 at a time?

33. What is the total number of the arrangements, without repetitions, that can be made with 16 objects, taking 1, 2, 3 ..... 16 at a time?

34. What will be the answer to the preceding problem if the arrangements are made with repetitions?

35. What is the total number of the arrangements, with repetitions, of 26 letters taken 2, 3, 4 ..... 10 at a time?

36. The number of arrangements, with repetitions, of n objects taken 8 at a time is to the number of arrangements, with repetitions, of the same objects taken 3 at a time, as 537,824 is to 1. Find n.

37. The total number of the arrangements, 1, 2, 3 ..... 8 at a time, of p objects, is to the total number of the arrangements 1, 2, 3, 4 at a time, as 1297 is to 1. In the two cases there are repetitions. Find p.

COMBINATIONS.

Suppose m letters are taken 2, 3, 4 ..... n at a time, and from them groups are formed in which the *order* of the elements is indifferent, but such that any two groups whatever differ from each other at least by one letter, the results that are obtained are *combinations* or *different products*.

If C_n is the number of combinations of m letters taken n at a time, we have

$$C_n = \frac{m(m-1)(m-2)\ldots\ldots(m-n+1)}{1\times 2\times 3\ldots\ldots n}.$$

If $\Sigma C_{1,2,3\ldots\ldots m}$ is the sum of all the combinations of m objects taken 1, 2, 3 ..... m at a time, we have

$$\Sigma C_{1,2,3\ldots\ldots m} = 2^m - 1.$$

If we have m collections of objects, containing respectively p, q, r, s ..... different objects, and if we make combinations, taking each time *only one* object in each collection, we have for the total number C of the combinations,

$$C = p\times q\times r\times s\ldots\ldots$$

If two collections contain respectively m and n objects, the number C of the combinations that can be obtained by taking p objects of the first and q objects of the second collection is given by the formula,

$$C = \frac{m(m-1)(m-2)\ldots\ldots(m-p+1)}{1\times 2\ldots\ldots p}\times\frac{n(n-1)(n-2)\ldots\ldots(n-q+1)}{1\times 2\ldots\ldots q}.$$

38. How many different products can be formed with the natural numbers, from 2 to 13 inclusive, multiplying 3 and 3?

39. A captain, having under his command 60 men, wishes to form a guard of 8 men. In how many different ways can it be formed?

40. A merchant who has 8 kinds of coffee wishes to make mixtures of equal parts, taking for each mixture 3 different qualities. How many different mixtures can he make?

41. To form a floor, a man has hexagonal bricks of 12 different colors. Arranging the colors 4 and 4, how many combinations of colors can be obtained?

42. An assembly containing 50 members wishes to elect a commission of 12 members taken from its midst. Among how many combinations can the assembly choose?

43. A number has for prime factors 2, 3, 7, 13, 17. How many divisors has this number, including the prime factors? The number itself and unity are not counted.

44. The actual number of 4 companies is reduced respectively to 40, 62, 70, and 73 men. In how many ways can an officer form a picket of 4 men, taking 1 soldier from each company?

45. In an election there are 6 candidates for 4 offices. If each elector can have upon his vote any number of names whatever, without exceeding 4, in how many ways can he vote?

46. It is wished to divide 15 objects in lots, each containing 3 objects. In how many ways can the lots be made?

47. A man has 5 pairs of pants, 8 waistcoats, and 7 coats. In how many different costumes can he appear?

48. How many different compounds will 66 simple bodies form, uniting 2 and 2, 3 and 3, 4 and 4, in the proportions of an atom of one element for an atom of the other?

49. How many combinations will these same elements form combining 2 and 2, 3 and 3, 4 and 4, if each element enters in the compounds for 1, 2, 3, or 4 atoms?

50. A detachment of 30 men must furnish each night a guard of 4 men. For how many nights can a different guard be formed, and how many times will each soldier serve?

51. How many combinations can be made with 10 letters, a, b, c, d ....., taken 5 at a time? In how many combinations will a enter? In how many will a, b, c enter at a time?

52. We have m different objects. Find the value of n, so that, combining these m objects n at a time, the number of the combinations is a maximum.

53. The number of combinations of $n+2$ objects, taken 4 at a time, is to the number of combinations of n objects, taken 2 at a time, as 11 is to 1. Find n.

54. The total number of the combinations of $2n$ objects, taken 1, 2, 3 together, is to the number of n objects as 387 is to 1. Find n.

55. Find how many divisors a number has which is equal to $2^5 \times 3 \times 5^4 \times 7 \times 13$.

56. How many divisors has the number
$$2^3 \times 5^2 \times 7 \times 11 \times 13?$$

57. How many divisors has the number
$$2^4 \times 3^5 \times 5^2 \times 7 \times 11 \times 13?$$

58. Prove that the number of combinations of m objects, taken n at a time, is the same as that of the combinations taken $(m-n)$ at a time.

59. It is required to divide 52 cards in 4 packs of 13 cards each. How many years will it take to form all the combinations possible, forming 4 combinations per minute, and working every day, 15 hours per day? The year being counted as 365 days and 6 hours.

60. An editor wishes to publish for youth a volume embellished with 12 chromos and 18 engravings. It is necessary to choose among 30 chromos and 37 engravings. How many different combinations can he make to illustrate the volume?

61. An assembly of stockholders, composed of 40 merchants, 20 lawyers, 30 manufacturers, and 10 physicians, wishes to elect a commission of 4 merchants, 3 manufacturers, 1 physician, 2 lawyers. In how many ways can the commission be formed?

Ex. 57.

Binomial Theorem.

If m is an entire positive quantity, we have (A):

$$(a+b)^m = a^m + \frac{m}{1}a^{m-1}b + \frac{m(m-1)}{1\times 2}a^{m-2}b^2$$
$$+ \frac{m(m-1)(m-2)}{1\times 2\times 3}a^{m-3}b^3 + \ldots\ldots \frac{m(m-1)\ldots\ldots 2}{1\times 2\times 3\times (m-1)}ab^{m-1}$$
$$+ \frac{m(m-1)\ldots\ldots 2\times 1}{1\times 2\times 3\ldots\ldots m}b^m.$$

The nth term $= \dfrac{m(m-1)(m-2)\ldots\ldots(m-n+2)}{1\times 2\times 3\ldots\ldots(n-1)}a^{m-n+1}b^{n-1}$.

Apply the formula to the development of the following expressions:

1. $(a+x)^3$.
2. $(b+x)^4$.
3. $(d+y)^5$.
4. $(c+z)^6$.
5. $(a+p)^7$.
6. $(m+n)^{11}$.

7. $(p+v)^{13}$.
8. $(a-b)^4$.
9. $(a-b)^5$.
10. $(a-x)^7$.
11. $(h+z)^{13}$.
12. $(h-z)^{11}$.
13. $(a-c)^8$.
14. $(b-p)^9$.
15. $(z-d)^{13}$.
16. $(1+x)^{12}$.
17. $(1-x)^9$.
18. $(1-a)^8$.
19. $(1+a)^{15}$.
20. $(1-z)^{14}$.
21. $(x+1)^9$.
22. $(x-1)^{11}$.
23. $(y-1)^8$.
24. $(z+1)^7$.
25. $(z-2)^8$.
26. $(2+x)^3$.
27. $(2-x)^4$.
28. $(3+x)^8$.
29. $(x-3)^9$.
30. $(z+5)^6$.
31. $(a-5)^8$.
32. $(4-b)^7$.
33. $(h-2)^{10}$.
34. $(a-10)^6$.
35. $(b+4)^5$.
36. $(\frac{1}{2}+x)^5$.
37. $(y+\frac{1}{3})^6$.
38. $(\frac{1}{4}-v)^7$.
39. $(a-\frac{1}{5})^8$.
40. $(0.3+x)^6$.
41. $(0.4-z)^7$.
42. $(3.2+b)^5$.
43. $(c+1.5)^4$.
44. $(a-2.1)^6$.
45. $(2a+x)^5$.
46. $(3a-y)^7$.
47. $(\frac{1}{2}a+z)^6$.
48. $(\frac{1}{3}b-y)^8$.
49. $(\frac{2}{3}x+y)^9$.
50. $(\frac{3}{4}a-b)^{11}$.
51. $(x-\frac{2}{5})^4$.
52. $(-x+2a)^7$.
53. $(-3x+b)^8$.
54. $(-y+2c)^9$.
55. $(-z+3a)^5$.
56. $(2x+3a)^6$.
57. $(5d-3y)^5$.
58. $(4d-3z)^7$.
59. $(0.2a+0.3b)^5$.
60. $(1.2b-0.2y)^7$.
61. $(\frac{1}{3}a+\frac{1}{2}b)^5$.
62. $(\frac{1}{5}c-\frac{1}{4}d)^8$.
63. $(\frac{2}{5}m+\frac{3}{4}p)^9$.
64. $(\frac{2}{5}z-\frac{3}{7}b)^6$.
65. $(\frac{3}{5}h-\frac{1}{2}v)^7$.
66. $(a^2+b^3)^5$.
67. $(x^3-a^5)^6$.
68. $(y^2+b)^7$.
69. $(d^3-b^2)^8$.
70. $(a^3-y^2)^9$.
71. $(d^3-3x^2)^8$.
72. $(2h^2-3x^3)^5$.
73. $(3a^2+5b^3)^7$.
74. $(1-2x^3)^5$.
75. $(3x^2-1)^8$.
76. $(1+\frac{2}{3}a^5)^8$.
77. $(\frac{1}{2}p^5+3y^4)^5$.
78. $(\frac{2}{3}a^3+\frac{3}{4}b^2)^6$.
79. $(0.2a^3-\frac{2}{3}x^4)^4$.
80. $(\frac{5}{3}l^2+0.3p^3)^7$.
81. $(2a-3cx)^7$.

82. $(3a^2+2c^3x^3)^6$.

83. $(bc^2-2ay^3)^5$.

84. $(4c^4-3acx^5)^4$.

85. $(a^2b^3+2a^3bx^4)^5$.

86. $(\frac{2}{3}ab^2-\frac{3}{4}a^3y)^7$.

87. $(\sqrt{a}+x)^7$.

88. $(\sqrt{2b}-m)^3$.

89. $(\sqrt{3c}+2a)^7$.

90. $(\sqrt{\frac{1}{2}a}-3y)^8$.

91. $(2\sqrt{e}+a)^6$.

92. $(2b-\sqrt{x})^9$.

93. $(\frac{2}{3}a+\sqrt{2x})^6$.

94. $(\frac{3}{4}a-\sqrt{\frac{1}{2}x})^5$.

95. $(2a-3\sqrt{y})^6$.

96. $(a^2+\frac{1}{2}\sqrt{z})^7$.

97. $(\sqrt{a}+\sqrt{x})^9$.

98. $(\sqrt{b}-\sqrt{y})^8$.

99. $(\sqrt{2c}+\sqrt{3x})^6$.

100. $(\sqrt{\frac{1}{2}a}-\sqrt{2x})^7$.

101. $(\sqrt{\frac{2}{3}a}+\sqrt{\frac{3}{4}x})^6$.

102. $(\sqrt{a^3}-\sqrt{x^5})^{10}$.

103. $(\sqrt{b^5}+\sqrt{y^4})^9$.

104. $(\sqrt{\frac{2}{3}h^3}+\sqrt{\frac{1}{2}x^4})^6$.

105. $(\sqrt[3]{2a}+\sqrt[3]{3x^2})^8$.

106. $(2ax+\sqrt[3]{3xy^2})^5$.

107. $(\sqrt{3ay}+\sqrt[3]{2x^2y})^7$.

108. $(3abx^3-\sqrt[3]{5b^2xy^2})^8$.

109. $\left(\frac{\sqrt{x}}{a}+\frac{\sqrt[3]{y}}{b}\right)^{10}$.

110. $\left(\frac{\sqrt[3]{x^2}}{a^2}+\frac{\sqrt[4]{xy^3}}{b}\right)^7$.

111. $\left(\frac{\sqrt{xz}}{y}-\frac{\sqrt[3]{ay^2}}{x^2}\right)^6$.

112. $(2a-x\sqrt{-1})^7$.

113. $(3b^2+2x\sqrt{-1})^6$.

114. $(\sqrt{a^3}+i\sqrt{x})^5$.

115. $(\sqrt[3]{a^2}+i\sqrt[4]{x^3})^6$.

116. $(\sqrt[3]{a^2}-\sqrt{-1})^9$.

117. $(i\sqrt{b}+i)^{10}$.

118. $(i\sqrt[3]{x^2}-i\sqrt{2})^9$.

119. $(i\sqrt[5]{y^4}+i\sqrt[3]{5})^7$.

120. $\left(\frac{2i\sqrt{b^3}}{y}-\frac{di\sqrt[3]{y^2}}{b}\right)^8$.

Find, in the following developments, the term demanded:

121. The seventh term of $(2+a)^{16}$.

122. The eleventh term of $(a+d)^{21}$.

123. The sixth term of $(3+2x^2)^9$.

124. The twelfth term of $(a^2 - 3x^5)^{20}$.

125. The eighteenth term of $(1 + x^2)^{30}$.

126. The fourteenth term of $(y^3 - 1)^{40}$.

127. The seventh term of $(\frac{1}{2}a - \sqrt{x^3})^{17}$.

128. The fourth term of $(\sqrt{a} - \sqrt[3]{x^2})^{11}$.

129. The sixth term of $(\sqrt{a} - ix)^{13}$.

130. The thirtieth term of $(b - 2c^2)^{50}$.

131. Prove that the coefficient of the nth term from the beginning is equal to the coefficient of the nth term from the end.

132. What is the greatest coefficient in the development of $(1 + x)^m$?

133. What is the greatest coefficient of the development of i. $(a + b^3)^{21}$? ii. $(b + x^2)^{16}$? iii. $(1 + y^5)^{32}$?

134. Prove that, in the development of $(1 + a)^m$, the sum of the coefficients of even places is equal to the sum of the coefficients of odd places.

135. Prove that, in the development of $(1 + x)^m$, the sum of the coefficients of the terms in x is equal to $2m - 1$.

Instead of $(a + b)^m$ we can write $a^m\left(1 + \frac{b}{a}\right)^m$; or, replacing in the parenthesis, $\frac{b}{a}$ by x, we can develop a quantity of the form $(1 + x)^m$. After the preceding formula (A), we have (B):

$$(1 + x)^m = 1 + \frac{m}{1}x + \frac{m(m-1)}{1 \times 2}x^2 + \frac{m(m-1)(m-2)}{1 \times 2 \times 3}x^3$$
$$+ \frac{m(m-1)(m-2)(m-3)}{1 \times 2 \times 3 \times 4}x^4 + \ldots\ldots$$

If m is a negative integral number, or a positive or negative fraction, the number of the terms of the second member of (B) is unlimited. If, in absolute value, $x > 1$, this second member increases

indefinitely, and cannot be equal to the first member, whose value is finite.

In the development of $\frac{1}{1-x}$, we have

$$\frac{1}{1-x} = (1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots\ldots$$

When $x = 2$, the first member is negative, while the second is positive and indefinitely increasing.

If, in absolute value, $x < 1$, the second member converges towards a fixed finite value, which is the value of the first member.

Then $(a+b)^m$ can be developed after the formula (A), where m is integral and negative, or fractional. Thus, if $a > b$, (C):

$$(a+b)^m = a^m\left(1+\frac{b}{a}\right)^m = a^m + \frac{m}{1}a^{m-1}b + \frac{m(m-1)}{1\times 2}a^{m-2}b^2 + \frac{m(m-1)(m-2)}{1\times 2\times 3}a^{m-3}b^3 + \ldots\ldots$$

If $a < b$, it will be developed thus, (D):

$$(a+b)^m = b^m\left(1+\frac{a}{b}\right)^m = b^m + \frac{m}{1}b^{m-1}a + \frac{m(m-1)}{1\times 2}b^{m-2}a^2 + \ldots\ldots$$

136. Develop $(1+x)^{-1}$, and then make:
$x = \frac{1}{9}$; $x = 0.07$; $x = -0.009$.

137. Develop $(1+x)^{\frac{1}{2}}$, and then make $x = \frac{3}{8}$.

138. Develop $(1+x)^{-\frac{1}{2}}$, and then make $x = 0.003$.

139. Develop $(1+x)^{\frac{1}{3}}$, and then make:
$x = \frac{7}{12}$; $x = -0.09$; $x = \frac{1}{8}$.

140. Develop $(1+x)^{-\frac{1}{3}}$, and then make:
$x = -\frac{2}{11}$; $x = -0.002$; $x = -\frac{1}{8}$.

141. Develop $(1-x)^{\frac{2}{3}}$, and then make $x = 0.004$.

Develop the following:

142. $(1+15)^{\frac{1}{4}}$.

143. $(2+27)^{\frac{1}{3}}$.

144. $(1+9)^{-\frac{1}{2}}$.

145. $(1+64)^{\frac{1}{3}}$.

4. A picture was raffled for. I took 5 shares. What was my expectation if there were 20 shares taken by other subscribers?

5. What is the chance of throwing 5 with one die? Also find the chance of throwing with two dice 3, 4, or 12; and, lastly, find the chance of throwing with two dice either 7, or else 8, or else 9.

6. There are three events, A, B, C, one of which must happen. The odds are 3 to 7 on A, and 3 to 6 on B. Find the odds on C.

7. What are the chances, for and against, that the number 7 will come in 3 throws of 2 dice?

8. The odds in one case being $2 : 2\frac{1}{2}$, in another $3\frac{1}{4} : 4$; compare the corresponding chances.

9. In a bag are 3 white, 4 red, and 5 black balls. If 2 balls be drawn, find the chance of their being (i.) white and red; (ii.) both black; (iii.) one at least red.

10. Five balls are drawn from a bag of 12, of which 3 are marked ones. Find the chance that among the 5 all the marked ones will be drawn.

11. Out of an urn which contains 4 white, 5 red, and 3 blue balls I take, blindfold, 3 balls. Required the probability that these will all be of different colors.

12. There are two bags, one containing 5 white balls and 2 black; the other containing 7 white balls and 3 black. If one be drawn from each bag, what is the probability of drawing (i.) a white ball from both? (ii.) a white ball and a black from the second? (iii.) a white ball from the second and a black from the first? (iv.) a black ball from both? (v.) one white ball? (vi.) at least one white ball?

13. The dealer in a whist party desires to know the probability of his partner holding two given cards.

14. A boy who has 7 marbles plays with me at "odd or even." Show the chance of my winning "even" is to the chance of my winning "odd" as 63 to 64. What with 5 balls?

15. In a bag there are 7 white, 5 red, 9 blue, and 14 yellow balls. If I draw out two, how much greater is the probability that they will be white and blue, rather than red and yellow?

16. Out of a mutinous crew of 120, the captain ordered 2 men, chosen by lot, to be executed, the real conspirators being 20. What is the chance that (i.) one of them, (ii.) two of them, will be taken?

17. In a bag there are 11 balls: 3 white and 8 red. If I draw 3 balls one after the other, find the chance of their being all white ones; also, the chance if I replace the ball drawn each time.

18. A bag contains 4 white and 7 black balls, and I draw one at a time. Find the chance that, after 3 trials, I shall have drawn 2 white and 1 black (i.) without replacing the ball drawn, (ii.) replacing each time the ball drawn.

19. If I toss a shilling 5 times, what are the odds against its falling "heads" 3 times?

20. A bag contains 4 white and 6 red balls. If I draw 4 times, replacing, show that 2 red or 3 red are events equally possible.

21. If 5 shillings are thrown into the air, show that it is just as likely that the event will show 3 heads and 2 tails as 2 heads and 3 tails.

22. In a bag are 5 black balls and 1 white. If I draw 3 times (replacing), what is the chance that I shall have drawn (i.) 3 black balls? (ii.) 3 white balls? (iii.) 2 black and 1 white?

23. With 2 dice, what is the probability of making 9 at the first throw? or, if this does not happen, 9 with the second throw?

24. A bag contains 7 balls: 5 white and 2 black. I draw a ball, receiving a shilling if it be a white one, but paying 3 pence if it be black. What is my expectation?

25. With 3 dice, what are the chances of throwing at the first time 3, 4, 5, and all numbers up to 18? and show that the odds are 1 : 7 against the two most likely numbers, namely, 10 and 11.

26. There are in a bag 4 balls: one is known to be white, and the others are either all white or else all black. A ball is drawn and is found to be white, and replaced. If, then, another is drawn, what is the chance of its being a black ball? If two are drawn, what is the chance that one will be a black ball and one a white ball?

Ex. 59.

Continued Fractions.

Change into continued fractions:

1. $\frac{103}{21}$.
2. $\frac{37}{11}$.
3. $\frac{89}{12}$.
4. $\frac{269}{18}$.
5. $\frac{67}{167}$.
6. $\frac{103}{643}$.
7. $\frac{149}{809}$.
8. $\frac{879}{583}$.
9. $\frac{484}{457}$.
10. $\frac{368}{103}$.
11. $\frac{167}{907}$.
12. $\frac{811}{991}$.
13. 9.749.
14. 5.07.
15. 0.0241.

16. $\dfrac{15a^3+1}{15a^3+6a}$.

17. $\dfrac{63m^5+1}{315m^6+9m^3+5m}$.

18. $\dfrac{x^3+x^2+x}{x^3+2x^2+x+1}$.

19. $\dfrac{24x^3+6x}{24x^4+18x^2+1}$.

20. $\dfrac{x^{10}+x^7+x^5+x^3+1}{x^9+x^4+x^2}$.

21. $\dfrac{a^3x^9+x^4+a^7x^2}{a^6x^{10}+ax^7+a^3x^5+a^5x^3+1}$.

Change to common fractions:

22. $\cfrac{1}{2+\cfrac{1}{3+\frac{1}{7}}}$.

24. $\cfrac{1}{8+\cfrac{1}{7+\frac{1}{3}}}$.

26. $2+\cfrac{1}{3+\cfrac{1}{5+\frac{1}{7}}}$.

23. $\cfrac{1}{6+\cfrac{1}{4+\frac{1}{2}}}$.

25. $\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\frac{1}{2}}}}$.

27. $4+\cfrac{1}{2+\cfrac{1}{4+\frac{1}{6}}}$.

28. $3+\cfrac{1}{2+\cfrac{1}{4+\frac{1}{8}}}$.

31. $5+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{4+\frac{1}{5}}}}}$.

29. $7+\cfrac{1}{6+\cfrac{1}{5+\frac{1}{4}}}$.

32. $\cfrac{1}{13+\cfrac{1}{11+\cfrac{1}{9+\cfrac{1}{7+\frac{1}{5}}}}}$.

30. $1+\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{1+\frac{1}{2}}}}$.

33. $\cfrac{1}{5+\cfrac{1}{7+\cfrac{1}{9+\cfrac{1}{11+\frac{1}{13}}}}}$.

Form the convergents of the continued fractions:

34. $\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{4+\cfrac{1}{5+\frac{1}{6}}}}}$.

35. $\cfrac{1}{3+\cfrac{1}{1+\cfrac{1}{4+\cfrac{1}{1+\frac{1}{5}}}}}$.

36. $\cfrac{1}{2+\cfrac{1}{7+\cfrac{1}{3+\cfrac{1}{9+\frac{1}{4}}}}}$.

37. $\cfrac{1}{4+\cfrac{1}{8+\cfrac{1}{16+\cfrac{1}{2+\frac{1}{4}}}}}$.

38. $\cfrac{1}{1+\cfrac{1}{3+\cfrac{1}{5+\cfrac{1}{7+\cfrac{1}{9+\frac{1}{11}}}}}}$.

39. $\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\frac{1}{3}}}}}}$.

Find the approximate values of:

40. $\frac{3845}{997}$.

41. $\frac{977}{4636}$.

42. $\frac{15600}{839}$.

43. $\frac{3791}{2951}$.

44. $\frac{13733}{18983}$.

45. $\frac{36661}{43523}$.

46. $\frac{2703}{2219}$.

47. $\frac{20467}{44341}$.

48. $\frac{3.1099}{36.391}$.

49. $\frac{48.427}{45.607}$.

50. 3.1415926.

51. 2.71828183.

52. 0.4342945.

53. $\frac{418\frac{4}{5}}{319.2}$.

Change to continued fractions:

54. $\sqrt{11}$. 56. $\sqrt{17}$. 58. $\sqrt{37}$. 60. $\sqrt{39}$.

55. $\sqrt{15}$. 57. $\sqrt{27}$. 59. $\sqrt{51}$. 61. $\sqrt{80}$.

Find the radicals which are equivalent to the following continued periodic fractions:

62. $\cfrac{1}{5+\cfrac{1}{5+\frac{1}{5}}}\cdots\cdots$

63. $\cfrac{1}{4+\cfrac{1}{4+\cfrac{1}{4+\frac{1}{4}}}}\cdots\cdots$

64. $\cfrac{1}{3+\cfrac{1}{3+\frac{1}{3}}}\cdots\cdots$

65. $1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\frac{1}{2}}}}\cdots\cdots$

66. $\cfrac{1}{1+\cfrac{1}{1+\frac{1}{1}}}\cdots\cdots$

67. $\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{1+\frac{1}{1}}}}\cdots\cdots$

68. $\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{2+\frac{1}{3}}}}\cdots\cdots$

69. The English yard is equivalent to 0.914383^{m}. What are the approximate ratios of the meter to the yard?

70. The English fathom is equivalent to 1.828767^{m}. What are the approximate ratios of the fathom to the meter?

71. What simple ratios express the value of the twenty-dollar gold piece, which weighs 33.437 grams, to that of the twenty-franc piece, which weighs 6.45161 grams, the two pieces being at the same standard?

72. If the equatorial radius of the earth is 6,377,398^{m}, and the polar radius is 6,356,080^{m}, what are the simple ratios of the equatorial diameter to the polar diameter?

73. The length of the tropical year is 365.242264 mean solar days. What are the simple ratios of this year with the civil year of 365 days?

Ex. 60.

INDETERMINATE COEFFICIENTS.

1. Find 4 terms of the quotient $\frac{1-2x}{1+4x}$.

2. Develop $\frac{1}{3-x}$.

3. Expand $\frac{1}{3-4a}$ to 5 terms.

4. Convert $\frac{1-x}{2+3x}$ into a series.

5. Expand $\frac{1}{1-2x+x^2}$.

6. Convert $\frac{1-x}{1+x-x^2}$ into a series.

7. Expand $\frac{1+x}{1-2x-x^2}$.

8. Develop $\sqrt{1+x^2}$.

9. Find the expansion of $\frac{a}{b-cx}$.

10. Find 4 terms of $\sqrt{a^2+b^2}$.

11. Expand $\frac{1}{2x-3x^2}$.

12. Convert $(1+x)^{\frac{1}{2}}$ into a series.

13. Expand $\frac{mx}{m^2-mx+x^2}$.

14. Develop $\frac{x(x-1)}{(x+1)^2}$ in a series.

15. Resolve $\frac{2x+3}{(x+2)(x-1)}$ into partial fractions.

16. Resolve $\frac{27+34x}{(3+4x)(6+7x)}$ into partial fractions.

17. Separate $\frac{1}{(x+1)(x+2)(x+3)}$ into partial fractions.

18. Separate $\frac{306x^3-450x+162}{(8x-7)(5x-4)(2x-1)}$ into partial fractions.

19. Resolve $\frac{P+Qx}{(1+ax)(1+bx)}$ into partial fractions.

20. Separate $\frac{a-bx}{(c-dx)(e-fx)}$ into partial fractions.

21. Resolve $\frac{4x-7}{x^2-x-6}$ into partial fractions.

22. Resolve $\frac{x^2+x+1}{(x-1)(x-2)(x-3)}$ into partial fractions.

23. Separate $\frac{x^2-x+1}{x^2(x-1)^2}$ into partial fractions.

24. Separate $\frac{2x^2-1}{x^3+1}$ into partial fractions.

25. Separate $\frac{3x-2}{x^2(x^2+1)^2}$ into partial fractions.

26. Resolve $\frac{x+a}{x(x-a)^2}$ into partial fractions.

Ex. 61.

Differential Method.

1. Required the fourth order of differences in the series, 1, 16, 81, 256, 625, 1296, 2401.....
2. Find the first term of the fourth order of differences in the series, 1^3, 2^3, 3^3, 4^3.....
3. Find the first term in the third order of differences in the series, 1, 5, 15, 35, 70.....; also, in the eighth order of 1, 3, 3^2, 3^3, 3^4, 3^5.....
4. Find the thirteenth term and the nth term of the series, 1, 4, 9, 16.....
5. Required the general term and the twelfth term of the series, 2, 6, 12, 20, 30.....
6. Calculate the thirtieth term of 1, $\frac{1}{3}$, $\frac{1}{6}$, $\frac{1}{10}$, $\frac{1}{15}$, $\frac{1}{21}$, $\frac{1}{28}$.....
7. Sum the series, 1^2, 2^2, 3^2, 4^2, 5^2, to ten terms, and also to n terms.
8. Sum to eighteen terms the series, 5, 9, 16, 26, 39.....
9. Find the sum of the series, 1, 3, 6, 10, 15, 21, continued to twelve terms, and also to n terms.
10. Sum $1^4 + 2^4 + 3^4 + 4^4 + 5^4 + \ldots\ldots + n^4$.
11. Given $\log 510 = 2.707570$, $\log 511 = 2.708420$, $\log 513 = 2.710117$, $\log 514 = 2.710963$; to find $\log 512$.
12. Given the cube roots of 45, 46, 47, and $49 = 3.5568933$, 3.5830479, 3.6088261, and 3.6593057; to find the cube root of 48.
13. Given $\log \tan 68° 54' = 10.413561$, $68° 55' = 10.413938$, $68° 56' = 10.414314$, $68° 57' = 10.414691$; to find $\log \tan 68° 56' 20''$.

14. Write down the general term, and sum *ad infinitum* the series, $\frac{1}{1\times 3}+\frac{1}{2\times 4}+\frac{1}{3\times 5}+\frac{1}{4\times 6}+\cdots\cdots$

15. Find the general term, and sum the series

$$\frac{1}{1\times 1}+\frac{1}{1\times 3}+\frac{1}{2\times 3}+\frac{1}{2\times 5}+\frac{1}{3\times 5}+\frac{1}{3\times 7}+\frac{1}{4\times 7}+\cdots\cdots, \textit{ ad infinitum.}$$

16. Sum *ad infinitum* the series whose general term is $\frac{1}{x(x+3)}$.

17. Find the general term, and sum *ad infinitum* $\frac{1}{1\times 5}+\frac{1}{5\times 9}+\frac{1}{9\times 13}+\cdots\cdots$

18. Sum *ad infinitum* the series $\frac{6}{2\times 7}+\frac{6}{7\times 12}+\frac{6}{12\times 17}+\frac{6}{17\times 22}+\cdots\cdots$

19. Sum to n terms the series whose general term is $\frac{1}{3x(4+4x)}$.

20. Sum *ad infinitum* $1+8x+27x^2+64x^3+125x^4+\cdots\cdots$

21. Sum *ad infinitum* $\frac{1}{5\times 11}+\frac{1}{8\times 14}+\frac{1}{11\times 17}+\cdots\cdots$

22. Required the number of balls in an incomplete square pile of 17 courses; a side of the base containing 24.

23. Required the sum of the infinite series

$$\frac{1}{1\times 2\times 3\times 4}+\frac{1}{2\times 3\times 4\times 5}+\frac{1}{3\times 4\times 5\times 6}+\frac{1}{4\times 5\times 6\times 7}+\cdots\cdots$$

24. Sum $1^3 + 3^3 + 5^3 + 7^3 + \cdots$ to n terms.

25. Find the scale of relation and sum *ad infinitum*

$$1 + 3x + 7x^2 + 13x^3 + 25x^4 + 51x^5 + 103x^6 + \cdots$$

26. (i.) Sum *ad infinitum*, and find the nth term of the series

$$1 + 5x + 15x^2 + 31x^3 + 53x^4 + 131x^5 + \cdots$$

(ii.) How many balls are contained in an incomplete rectangular pile of 12 courses, the sides of the base containing 40 and 20?

Ex. 62.

Equations of Higher Degrees.

Find the equations which have the following roots:

1. 1, 2, 3.

2. 1, 1, -2.

3. 2, 3, -5.

4. 3, -4, -7.

5. 7, $+\sqrt{5}$, $-\sqrt{5}$.

6. 2, $+\sqrt{3}$, $-\sqrt{-3}$.

7. 3, $\frac{-3+\sqrt{5}}{2}$, $\frac{-3-\sqrt{5}}{2}$.

8. 2, 3, $\frac{1}{2}$, $\frac{1}{3}$.

9. -4, -3, $3 \pm \sqrt{5}$.

10. $1 \pm \sqrt{2}$, $\frac{1}{2}(1 \pm \sqrt{-1})$.

Equations Solved like Quadratics.

11. $x^4 - 13x^2 + 36 = 0$.

12. $x^4 - 21x^2 = 100$.

13. $10x^4 - 21 = x^2$.

14. $2x^{\frac{1}{2}} - 3x^{\frac{1}{3}} + x = 0$.

15. $x^2 + 5 = 8x + 2\sqrt{x^2 - 8x + 40}$.

16. $2x^2 + 3\sqrt{x^2 - x + 1} = 2x + 3$.

17. $2x^4 + 5x^2 = 21\frac{3}{8}$.

18. $\frac{x^6}{3} - \frac{x^3}{4} = 2\frac{1}{4}$.

19. $\frac{12x^3 - 11x^2 + 10x - 78}{8x^2 - 7x + 6} = 1\frac{1}{2}x - \frac{1}{2}$.

20. $(8x+3)^2+(8x+3)=30$.

21. $\sqrt[3]{x^2}+3\sqrt[3]{x}=1\frac{3}{4}$.

22. $\sqrt{x^5}+\sqrt{x^3}=6\sqrt{x}$.

23. $3x^{\frac{1}{2}}-2x^{\frac{1}{4}}=-20$.

24. $5x^{2n}+3x^n=6\frac{3}{4}$.

25. $2(x^2-x+1)-\sqrt{x^2-x+1}=\frac{1}{4}$.

26. $x^2-2x^{\frac{3}{2}}+2x-\sqrt{x}=6$.

27. $x^8-97x^4=-1296$.

28. $x^3-31x^{\frac{3}{2}}=-108$.

29. $(2.5-x)^4+0.5625=2.5(2.5-x)^2$.

30. $\sqrt{x^2-8x+31}+(x-4)^2=5$.

31. $\frac{8}{x^3}+2=17\div x^{\frac{3}{2}}$.

32. $2x^2+4x+3\sqrt{2x^2+4x+9}=31$.

33. $5x-7x^{\frac{1}{2}}+8\sqrt{7x^2-5x+1}=8$.

34. $x^{\frac{1}{2}}-40\div x^{\frac{1}{4}}=3x$.

35. $\frac{2x-\sqrt{2x+11}}{2x+\sqrt{2x+11}}=\frac{9}{19}$.

36. $\frac{x-\sqrt{x}}{x+\sqrt{x}}=\frac{4}{x^2-x}$.

37. $\frac{a+x+\sqrt{a^2-x^2}}{a+x-\sqrt{a^2-x^2}}=\frac{c}{x}$.

38. $9x+4+2x\sqrt{9x+4}=15x^2$.

39. $x^2-7=\sqrt{x^2-42x+89}$.

40. $\frac{x+\sqrt{x}}{x-\sqrt{x}}=\frac{3\sqrt{x}+6}{2\sqrt{x}}$.

Cubic and Reciprocal Equations.

Find the roots of:

41. $x^3 + 10x = 104$.
42. $x^3 - 9x^2 + 26x = 24$.
43. $x^3 - 5x = 12$.
44. $x^3 - 8x^2 + 5x = -14$.
45. $x^3 - 18x = -27$.
46. $x^3 - 18x^2 + 87x = 110$.
47. $x^3 - 5x = -12$.
48. $x^4 - 10x^3 + 35x^2 - 50x = -24$.
49. $x^3 + 2x = 135$.
50. $x^4 - 45x^3 - 40x = -84$.
51. $x^3 = 1$.
52. $x^3 - \frac{1}{8}(14x^2) + 7x = 3\frac{1}{8}$.
53. $x^3 + \frac{8x^2}{5} + \frac{9x}{100} = \frac{9}{100}$.
54. $x^3 - \frac{49x}{3} = \frac{524}{27}$.
55. $x^3 - \frac{13x^2}{12} + \frac{3x}{8} = \frac{1}{24}$.

Find one root of:

56. $x^3 + 6x = 2$, to five decimal places.
57. $x^3 - 6x = 12$, to three decimal places.
58. $x^3 + 9x = -6$, to four decimal places.
59. $x^3 + x^2 = 500$, to four decimal places.
60. $x^3 + 12x = 20$, to four decimal places.

Remove the second term from:

61. $x^3 - 6x^2 + 5x = 84$.
62. $x^2 + 6x = 91$.
63. $x^3 - 15x^2 + 81x - 243 = 0$.
64. $x^3 - 6x^2 + 18x - 22 = 0$.
65. $x^3 + 4x^3 - 5x - 24 = 0$.
66. $x^4 + 2x^3 - 7x^2 - 8x + 12 = 0$.

Find the three roots of:

67. $x^3 - 6x^2 + 10x - 8 = 0$.
68. $x^3 + 3x = 5$.

69. $x^3 - 6x^2 - 12x + 112 = 0.$

70. $x^3 + 12x^2 + 45x + 50 = 0.$

71. $x^3 - 21x^2 + 159x - 490 = 0.$

Solve the reciprocal equations:

72. $x^4 + 2x^3 + 2x + 1 = \frac{1}{9}(142x^2).$

73. $x^6 - 6x^5 + 14x^4 - 18x^3 + 14x^2 - 6x = -1.$

Solve by means of a table of sines:

74. $x^3 - 5x + 4 = 0.$

Solve by Horner's Method:

75. $x^3 - 9x = 9.$

Approximate by double position to a value of x in:

76. $x^3 + 10x^2 + 5x = 2600.$

Approximate, by a series of converging fractions, to a value of x in:

77. $x^3 + 7x + 3 = 0.$

Approximate, by a table of sines, to a value of x in:

78. $x^3 - 12x - 16 = 0.$

Ex. 63.

Exponential Equations.

Find x:

1. Given $6^x = 7776.$

2. Given $100^x = 250.$

3. Given $10^x = 2.7182818.$

4. Given $2.5^x = 34.56.$

5. Given $\left(\frac{1}{x}\right)^{\frac{1}{2.7182818}} = 0.6922006.$

6. Given $(-1.23)^x = -2.8153057$.

7. Given $2^{3x} \times 5^{2x-1} = 4^{5x} \times 3^{x+1}$.

8. Given $(2\frac{1}{4})^{3\frac{3}{8}} = (3\frac{3}{8})^x$.

9. Given $0.12345^{6x} = 1{,}697{,}365$.

10. Given $3^{(5x)} = 7$.

11. Given $(1\frac{2}{3})^{4+\frac{1}{3}x} = 151.884$.

12. Given $a^{mx+n} = b^{nx+m}$.

13. Given $\sqrt[x]{3^{5x+7}} = \sqrt[7x]{5^{3x+1}}$.

Find x and y:

14. Given $a^x a^{5y} = (a^7)^4$, and $a^{7x} \div a^6 = (a^y)^3$.

15. Given $a^x b^y = p$, and $c^x d^y = q$.

16. Given $\sqrt[x]{a} \times \sqrt[y]{a} = a^{\frac{7}{12}}$; $\sqrt[x]{a^3} \div \sqrt[y]{a^4} = 1$.

17. Given $a^x + a^{-x} = b$, to determine the value of x.

18. Given $x^x = 100$, to approximate to a value of x.

19. Given $x^x = 42.8454$, to approximate to a value of x.

20. Given $x^{\frac{1}{x}} = 1.2655$, to approximate to a value of x.

21. Given $x^y = 243$, and $\sqrt[y]{1024} = (\frac{2}{3}x)^2$, to find x and y.

22. Given $\sqrt[x]{x+y} = 2$, and $(x+y) \times 3^x = 279{,}936$, to find x and y.

23. Given $3^x \times 2^y = 72$; $2^x \times 6^z = 144$; $3^y \times 4^z = 432$, to find x, y, and z.

24. Given $10^{(5-x)(6-x)} = 100$, to find x.

25. Write out a proof of the Exponential Theorem.

26. Given $7^{\frac{x+1}{x+2}} = 6.70375^{\frac{x+3}{x+4}}$, to determine x.

[illegible]

Rule - Multiply or divide both members of the equation by a quantity that will make the first term a perfect square, then divide the coefficient of the second term by twice the square root of the coefficient of the first term and add the square of the quotient to both members.

The Hindoo Method - If the equation reduced to the form $(ax^2 + bx = c)$, be multiplied through by 4 times the coefficient of x^2, the quantity to be added to both members is always the square of the coefficient of x in the original equation. This method avoids fractions but gives rise to large numbers.

The Eclectic Method - [illegible] reduced to the form $ax^2 + (bx = c)$ be multiplied through by the coefficient of x^2, the quantity to be added to both members is always the square of $\frac{1}{2}$ the coeff. of x in the original equation. This method avoids fractions & large numbers when the co. of x is even the equation will remain integral [illegible]

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$k + 5y$
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